A MANUAL OF QUATERNIONS.

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BY

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PREFACE.

Readers of the Life of Sir William Rowan Hamilton will recollect that he undertook the publication of a book on quaternions to serve as an introduction to his great volume of *Lectures*. This *Manual of Quaternions* was intended to occupy about 400 pages, but while the printing slowly progressed it grew to such a size that it came to be regarded by its author as a "book of reference" rather than as a text-book, and the title was accordingly changed to *The Elements of Quaternions*. By a curious series of events one of Hamilton's successors at the Observatory of Trinity College has felt himself obliged to endeavour to carry out to the best of his ability Hamilton's original intention. And on the centenary of Hamilton's birth a Manual of Quaternions is offered to the mathematical world.

Last year I was called upon by the Board of Trinity College to assist in the examination for Fellowship. I had long ago recognized that another work on quaternions was required, and this want was forcibly brought home to me by my new duties. A mathematician, whose time is limited, is frightened at the magnitude of Hamilton's bulky tomes, although a closer acquaintance with the *Elements* would reveal the admirable lucidity and the logical completeness of that wonderful book, and although the *Lectures* have a charm all their own. The student wants to attain, by the shortest and simplest route, to a working knowledge of the calculus; he cannot be expected to undertake the study of quaternions in the hope of being rewarded by the beauty of the ideas and by the elegance of the analysis. And for his sake, though with reluctance I must confess, I have abandoned Hamilton's methods of establishing the laws of quaternions.

By a brilliant flash of genius Hamilton extended to vectors Euclid's conception of ratio. A quaternion is the mutual relation of two *directed* magnitudes with respect to *quantity and direction* as a ratio is the mutual relation of two undirected magnitudes with respect to *quantity*. From this enlarged view of a ratio, the calculus of quaternions is developed in the *Elements*. But the way is long and winding, and after much labour, I found I could not greatly shorten it or make it much less indirect. I therefore adopted another plan.

The two cardinal functions of two vectors are $S\alpha\beta$ and $V\alpha\beta$. These functions may be defined by the statements that $-S\alpha\beta$ is the product of the length of one vector into the projection of the other upon it, and that $V\alpha\beta$ is the vector which is perpendicular to α and to β, and which contains as many units of length as there are units of area in the parallelogram determined by α and β. Both these functions enjoy some of the properties of an algebraic product. They are distributive with respect to each of the vectors.

The *product* of the vector α into β may be defined to be the sum of these functions,

$$\alpha\beta = S\alpha\beta + V\alpha\beta.$$

This is a quaternion—the sum of a scalar and a vector. A product of a pair of vectors is distributive but not commutative. It is now necessary to define the product of a quaternion (q) into a vector (γ), and we say that it is the sum of the product of the scalar (Sq) into γ and the product of the vector (Vq) into γ, or that

$$q \cdot \gamma = Sq \cdot \gamma + Vq \cdot \gamma.$$

From these principles it follows almost immediately that quaternion multiplication is associative as well as distributive.

Division is seen to be deducible from multiplication, and on p. 12 we arrive at the important result that every function of quaternions formed by ordinary algebraic processes is a quaternion, scalars and vectors being considered to be special cases.

What we may call the grammar of the subject may be said to terminate on p. 20, the laws of combination of quaternions having been established, the five special symbols S, V, K, T and U having been defined and their chief properties explained, various constructions for products and quotients having been made, and the non-commutative property of multiplication having been illustrated by conical rotations and otherwise.

In the succeeding chapters, I have not scrupled to introduce, either in the articles in small type or in the worked examples in small type, illustrations of the applications of quaternions to subjects that can hardly be supposed to be familiar to the beginner in mathematics. It is suggested in the table of contents that these more difficult portions should be omitted by a beginner at first reading. The book is, however, primarily intended for those who commence the study of quaternions with a fair knowledge of other branches of mathematics; in other words, it is written for the majority of those at present likely to read quaternions because, as yet, the subject is not generally taught in elementary classes. On the other hand, I have abstained from printing examples of an artificial nature, and I have avoided unnecessary difficulties.

Although this book may be regarded as introductory to the works of Hamilton, it may also to some extent be considered as supplementing them. Many of the results contained in it have appeared only in the publications of learned societies, and many others are believed to be novel. It is possible, therefore, that this volume may be found to have some points of interest for the advanced student of quaternions. He will find, for example, that quaternions lend themselves to the treatment of projective geometry quite as readily as to investigations in mathematical physics and in metrical geometry.

By means of a somewhat elaborate table of contents, modelled on those prefixed by Hamilton to his *Lectures* and *Elements*, and by the aid of a full index and numerous cross references, I trust that the contents of this book will be found to be fairly accessible to the casual reader as well as to the systematic student. It must be remembered, however, that the objects of a work of this nature are to introduce a subject of the highest educational value, and to develop a powerful and comprehensive calculus. Such ends can be attained only by illustration and by suggestion, and it is not easy to tabulate methods of investigation.

It would be impossible to overestimate what I owe to Hamilton's *Lectures on Quaternions* (Dublin, 1853) and to his *Elements of Quaternions* (London, 1866, 2nd edition, in two volumes, with notes and appendices by C. J. Joly, London, 1899, 1901). The admirable *Elementary Treatise on Quaternions* (3rd edition, Cambridge, 1890), by the late Professor P. G. Tait—who has done so much for quaternions by his classical applications of Hamilton's operator ∇—has also been very useful. Other writers to whom I am indebted are referred to in the text.* I am glad to have this opportunity of offering my thanks to my respected friend, Benjamin Williamson, Esq., F.R.S., Senior Fellow of Trinity College, Dublin, for his great kindness in assisting me with a considerable portion of the proofs. I am also indebted to him for the uninterrupted encouragement he has given me, alike privately and in his official capacity as a member of the governing body of Trinity College, in my attempts to render Hamilton's work more widely known.

CHARLES JASPER JOLY.

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*The Bibliography by Dr. Macfarlane, published by the International Association for the promotion of the Study of Quaternions and Allied Systems of Mathematics (Dublin, 1904), renders unnecessary any detailed list of works on quaternions.

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† Articles 58-61 and possibly Arts. 55-56 may be omitted at first reading.

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CHAPTER I.

THE ADDITION AND SUBTRACTION OF VECTORS.*

ART. 1. A right line, AB, considered as having not only length but also direction, is said to be a vector. The direction of the vector AB is that of the point B as viewed from A, and the vector BA is the *opposite* of AB, being equal to it in length but having the opposite direction. All equal right lines AB, A′B′, etc., which have the same direction are equal vectors.†

ART. 2. The sum obtained by adding the vector BC to AB is denoted by BC+AB, and is defined to be the vector AC. Thus symbolically (fig. 1),

$$BC+AB=AC.$$

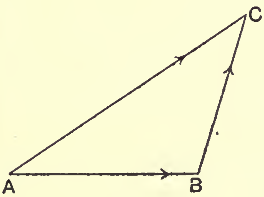

FIG. 1.

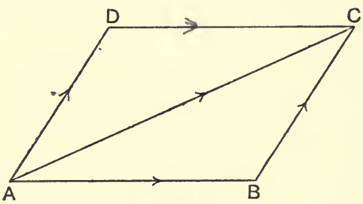

FIG. 2.

Completing the parallelogram, ABCD, the definition of addition gives likewise the equation (fig. 2)

$$DC+AD=AC$$

or

$$AB+BC=AC,$$

because the vectors DC and AD are respectively equal to AB and BC. Thus the sum of two vectors is independent of the order

* Following the example of Hamilton in his *Lectures on Quaternions* and in his *Elements of Quaternions*, the table of contents of this volume is amplified into an analysis or commentary to which it may be useful occasionally to refer.

† It seems to be an unnecessary complication to print a bar ($\overline{AB}$) over the letters which represent a vector AB. Hamilton sometimes uses the notation $\overline{AB}$ to represent the *length* of the vector AB.

in which they are added, or the addition of two vectors is a *commutative* operation.*

ART. **3**. The sum obtained by adding any vector CD † to the sum of AB and BC (fig. 3), is the sum of CD and AC, or the vector AD. But AD is likewise the sum of AB and BD, that is, the sum of AB and the sum of BC and CD. And by completing

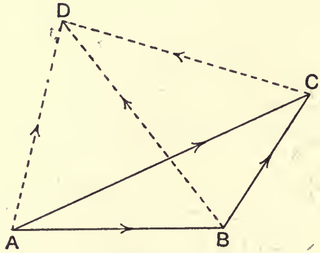

FIG. 3.

the parallelogram of which BD is a diagonal and BC and CD are sides, it appears that AD is also the sum of BC and the sum of AB and CD. In other words, the same vector is obtained by adding any one of the three vectors, AB, BC and CD, to the sum of the other two. This vector sum AD is consequently independent of the order in which the component vectors are taken and of the mode in which they are grouped.

The same process applies in general, and *the addition of vectors is an associative and a commutative operation.* It is associative inasmuch as the vectors may be grouped into partial sums in any way; and it is commutative because the order in which the vectors are taken is immaterial.

ART. **4**. Any number of vectors being arranged as the successive sides AB, BC, etc., of a polygon, their sum is the vector AD drawn from the initial point of the first to the terminal point of the last. If the polygon happens to be closed, the sum is a vector of zero length, or simply zero. Thus, in particular,

$$\mathrm{AB}+\mathrm{BA}=0, \quad \mathrm{AB}+\mathrm{BC}+\mathrm{CA}=0, \quad \mathrm{AB}+\mathrm{BC}+\mathrm{CD}+\mathrm{DA}=0.$$

ART. **5**. It is natural, in accordance with the equations just given, to introduce the sign $-$, and to write

$$\mathrm{BA}=-\mathrm{AB},$$

* In certain systems of vector analysis, the word vector is used in a different sense, and a vector cannot be determined without reference to its position. The commutative law then ceases to be obeyed. An example of non-commutative addition will be found in Art. 21, p. 16.

† In every case, unless the contrary is expressed or implied, the vectors with which we deal are not necessarily parallel to a plane.

or to agree that the sign $-$ prefixed to a vector shall convert it into its opposite (Art. 1). Hence the subtraction of one vector from another may be regarded as equivalent to the addition of the opposite of the first vector to the second. Subtraction of vectors is thus included in addition.

As we can now interpret $-\text{AB}$, it is convenient to use a single symbol to denote a vector. We shall follow Hamilton's admirable notation, and shall employ the small letters of the Greek alphabet to represent vectors, using, as a general rule, the earlier letters α, β, γ, etc., for given or constant vectors, and ρ or σ for variable vectors.

ART. **6**. The sum of two equal vectors is a vector of the same direction and of twice the length. It is natural to write, as in algebra,

$$2\alpha=\alpha+\alpha, \quad 3\alpha=\alpha+\alpha+\alpha, \quad \text{etc.},$$

and generally, at least when n is an integer,

$$\beta=n\alpha,$$

if the vectors β and α have the same direction while the length of β is n times that of α. This result may be extended to the case in which n is fractional or incommensurable by a process identical with similar extensions in elementary algebra. The last article affords the interpretation to be adopted when n is negative; and when n is complex $(n'+\sqrt{-1}n'')$, the difficulties of interpretation are of the same nature as in ordinary algebra, and need not be discussed here.

Further, it is natural to say that the coefficient n results from the division of the vector β by the parallel vector α, and we shall therefore write

$$n=\frac{\beta}{\alpha}, \quad \text{or} \quad n=\beta\div\alpha, \quad \text{or} \quad n=\beta:\alpha,$$

as a consequence of $\beta=n\alpha$. Also, conversely, whenever the quotient of two vectors is an algebraic quantity or a *scalar*,* we infer that the vectors are parallel, and that they have the same or opposite directions according as that scalar is positive or negative.

Again, if n is an integer and if α and β are any two vectors, the laws of addition give

$$n(\alpha+\beta)=n\alpha+n\beta,$$

and by a process of induction this relation may be extended to

* The word 'scalar,' synonymous with algebraic quantity, was employed by Hamilton because such a quantity may be conceived to be constructed by "comparison of positions upon one common scale (or axis)." *Elements*, Art. 17.

the case in which n is fractional or incommensurable. More generally, if x, y and z are any scalars,

$$z(x\alpha+y\beta)=zx\alpha+zy\beta,$$

so that the multiplication of vectors by scalars is a *distributive* operation.

ART. 7. In the calculus of quaternions a *unit of length* is selected to which the lengths of all vectors are referred. The tensor of a vector α is the number of units contained in its length, and is denoted by the symbol $\mathrm{T}\alpha$. Thus the tensor is a positive or "signless" number, at least when the vector is real,* and in particular,

$$\mathrm{T}\alpha=\mathrm{T}(-\alpha).$$

In general, if n is a real scalar,

$$\mathrm{T}n\alpha=n\mathrm{T}\alpha \text{ if } n>0; \quad \mathrm{T}n\alpha=-n\mathrm{T}\alpha \text{ if } n<0.$$

Hamilton also uses the notation $\mathrm{U}\alpha$ to denote a vector of unit length having the same direction as α, and he calls $\mathrm{U}\alpha$ the *versor* of the vector α. Since the direction of $-\alpha$ is opposite to that of α,

$$\mathrm{U}\alpha=-\mathrm{U}(-\alpha),$$

and, more generally,

$$\mathrm{U}n\alpha=\mathrm{U}\alpha \text{ if } n>0; \quad \mathrm{U}n\alpha=-\mathrm{U}\alpha \text{ if } n<0.$$

Also, by Art. 6,

$$\alpha=\mathrm{T}\alpha\,.\,\mathrm{U}\alpha,$$

or a vector is the product of its tensor and its versor.

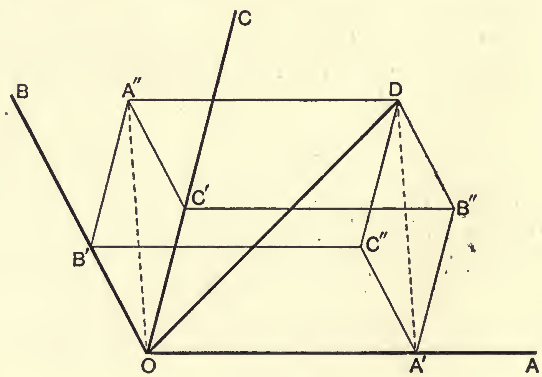

FIG. 4.

ART. 8. An arbitrary vector OD (or δ) may be resolved in one way into a sum of vectors parallel to three given and non-coplanar vectors OA, OB and OC (or α, β and γ).

* For imaginary vectors see Art. 22, p. 20.

Through D draw three planes parallel to the planes BOC, COA and AOB, meeting the lines OA, OB and OC in the points A′, B′, C′. Then it is evident from the figure that

$$\text{OD} = \text{OA}' + \text{OB}' + \text{OC}'; \quad \text{or} \quad \text{OD} = x\text{OA} + y\text{OB} + z\text{OC};$$
$$\text{or} \quad \delta = x\alpha + y\beta + z\gamma,$$

if the scalars x, y and z are the quotients of parallel vectors,

$$x = \text{OA}' : \text{OA}, \quad y = \text{OB}' : \text{OB}, \quad z = \text{OC}' : \text{OC};$$

and it is further evident that this construction is unique.

It may happen that some or all of these three scalars are negative, or some may be zero, but these cases can present no difficulty.

Ex. 1. Find the vector OC to a point which divides AB in a given ratio.

$$\left[\text{Here } \frac{\text{AC}}{m} = \frac{\text{CB}}{l} = \frac{\text{AC}+\text{CB}}{l+m} = \frac{\beta-\alpha}{l+m} = \frac{\gamma-\alpha}{m}, \text{ or } \gamma = \frac{l\alpha+m\beta}{l+m}.\right]$$

Ex. 2. If weights l, m and n are placed at A, B and C, find their centre of mass.

[The extremity of the vector $(l\alpha+m\beta+n\gamma):(l+m+n)$, supposed to be coinitial with α, β and γ.]

Ex. 3. Prove that the mean centre of a tetrahedron is (a) the intersection of bisectors of opposite edges; (b) the intersection of lines joining the vertices to the mean points of the opposite faces. Show that the former lines bisect one another, and that the latter quadrisect one another.

Ex. 4. Prove that the vectors $\pm\alpha\pm\beta\pm\gamma$ when drawn through a common point terminate at the vertices of a parallelepiped.

Ex. 5. Discuss the arrangement of the extremities of the sixteen coinitial vectors $\pm\alpha\pm\beta\pm\gamma\pm\delta$. Consider the points with reference to the extremities of $\pm\alpha$, etc., and with reference to one of the points, the extremity of $\alpha+\beta+\gamma+\delta$ for example.

Ex. 6. Prove that four arbitrary vectors are connected by a linear relation,

$$a\alpha+b\beta+c\gamma+d\delta=0.$$

Ex. 7. If three vectors are linearly connected, or if

$$a\alpha+b\beta+c\gamma=0,$$

they are coplanar.

Ex. 8. If $a\text{OA}+b\text{OB}+c\text{OC}=0$, $a+b+c=0$, the points A, B, C are collinear.

Ex. 9. If $a\text{OA}+b\text{OB}+c\text{OC}+d\text{OD}=0$, $a+b+c+d=0$, the points A, B, C, D are coplanar.

CHAPTER II.

MULTIPLICATION AND DIVISION OF VECTORS AND OF QUATERNIONS.

ART. **9.** The product of the length of one vector (α) into the length of the projection of another (β) upon it is denoted by the expression

$$-S\alpha\beta,$$

and this function $S\alpha\beta$ of two vectors is called the *scalar* of $\alpha\beta$.

By similar triangles it follows that (fig. 5)

$$S\alpha\beta = S\beta\alpha,$$

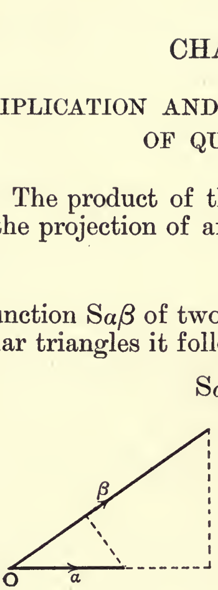

FIG. 5.

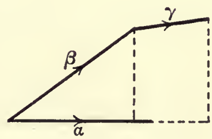

FIG. 6.

and because the sum of the projections of any number of vectors on any line is the projection of their sum, it appears that (fig. 6)

$$S\alpha(\beta+\gamma) = S\alpha\beta + S\alpha\gamma;$$

and therefore the function is a doubly distributive function, or

$$S\Sigma\alpha\Sigma\beta = \Sigma\Sigma S\alpha\beta.$$

If the vectors α and γ are at right angles,

$$S\alpha\gamma = 0,$$

and conversely.

An equation such as $\qquad S\alpha\beta = S\gamma\delta$

implies that the projection of α on β multiplied by the length of β is equal to the projection of γ on δ into the length of δ.

ART. **10**. A unit of length having been assumed, let a vector be drawn at right angles to two given vectors α and β so that rotation round this vector from α to β is positive,* and let the length of this vector be numerically equal to the area of the parallelogram determined by α and β. This vector is denoted by the symbol

$$\mathrm{V}\alpha\beta,$$

and is called the vector of $\alpha\beta$.

If the vectors are taken in the reverse order, $\mathrm{V}\beta\alpha$ has the same length as $\mathrm{V}\alpha\beta$, but the direction is opposite, the rotation being now reversed, so that

$$\mathrm{V}\beta\alpha = -\mathrm{V}\alpha\beta.$$

If an equation such as $\mathrm{V}\alpha\beta = \mathrm{V}\gamma\delta$

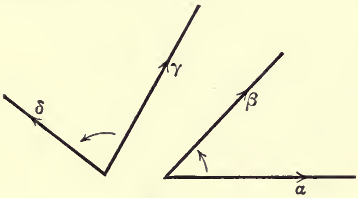

FIG. 7.

exists, the vectors α, β, γ and δ must all be parallel to the same plane; the areas of the parallelograms determined by α and β and by γ and δ must be equal, and the sense of rotation from α to β must be the same as that from γ to δ (fig. 7).

Like $\mathrm{S}\alpha\beta$, the function $\mathrm{V}\alpha\beta$ is a doubly distributive function. If β' is the component of the vector β at right angles to α it is obvious that

$$\mathrm{V}\alpha\beta = \mathrm{V}\alpha\beta',$$

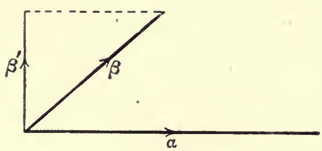

FIG. 8.

and the tensor of $\mathrm{V}\alpha\beta$ is equal to the product of the tensors of α and of β' (fig. 8).

* The convention respecting rotation which is here adopted is the opposite of that employed by Hamilton. The axis of a rotation is taken to be in the direction of the advance of a right-handed screw turning in a fixed nut, and this system is now known as the *right-handed* system of rotation (Clerk Maxwell, *Electricity and Magnetism*, Art. 23). On the other hand Hamilton calls his system right-handed, but he takes as the axis the direction from blade to handle of a turn screw when screwing a right-handed screw *into* a nut (*Lectures*, Art. 68, *Elements*, note to Art. 295), and accordingly some little care is necessary in comparing Hamilton's demonstrations with those of the present volume. Tait uses the modern right-handed system in his quaternion writings.

If β' and γ' are the components of β and γ at right angles to α, and in the plane of the paper while α is drawn upwards at right angles to the plane (fig. 9), the vectors $V\alpha\beta'$ and $V\alpha\gamma'$ will

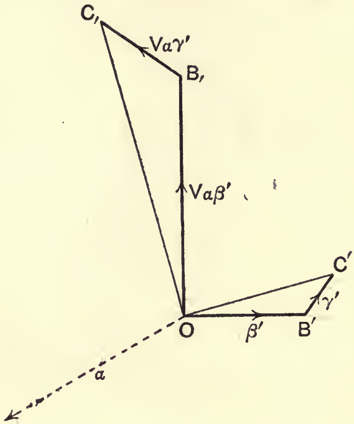

FIG. 9.

lie in the plane of the paper, at right angles respectively to β' and γ'. But $TV\alpha\beta' : T\beta' = TV\alpha\gamma' : T\gamma' = T\alpha$, and consequently the triangles $OB'C'$ and $OB_{,}C_{,}$ are directly similar. Hence $OC_{,}$ is at right angles to OC' and $TOC_{,} : TOC' = T\alpha$. Consequently

$$OC_{,} = V\alpha(\beta' + \gamma') = OB_{,} + B_{,}C_{,} = V\alpha\beta' + V\alpha\gamma'.$$

In this relation we may replace β' and γ' by β and γ, so that

$$V\alpha(\beta+\gamma) = V\alpha\beta + V\alpha\gamma\,;\; V(\beta+\gamma)\alpha = V\beta\alpha + V\gamma\alpha,$$

α, β, and γ being three arbitrary vectors.

We have now
$$V\Sigma\alpha\Sigma\beta = \Sigma\Sigma V\alpha\beta$$
for any number of vectors, since in particular for four vectors,

$$V(\alpha+\beta)(\gamma+\delta) = V(\alpha+\beta)\gamma + V(\alpha+\beta)\delta = V\alpha\gamma + V\beta\gamma + V\alpha\delta + V\beta\delta.$$

If $V\alpha\beta = 0$ without having either α or β zero, the vector α must be parallel to β, for the area of the parallelogram determined by α and β must vanish.

ART. **11.** The *product* of the vector α into β is defined by the equation,
$$\alpha\beta = S\alpha\beta + V\alpha\beta, \text{..........................} (B)$$
and because it is the sum of two doubly distributive parts, it is likewise doubly distributive, or

$$\Sigma\alpha\Sigma\beta = \Sigma\Sigma\alpha\beta.$$

The product $\beta\alpha$ is not generally equal to $\alpha\beta$. In fact

$$\beta\alpha = S\alpha\beta - V\alpha\beta \quad \text{because} \quad S\alpha\beta = S\beta\alpha,\; V\alpha\beta = -V\beta\alpha.$$

Thus multiplication of vectors is not commutative. We speak of $\alpha\beta$ as the product of β by α, or the product of α into β.

Adding and subtracting the expressions for the two products $\alpha\beta$ and $\beta\alpha$, we find

$$S\alpha\beta = \tfrac{1}{2}(\alpha\beta + \beta\alpha), \quad V\alpha\beta = \tfrac{1}{2}(\alpha\beta - \beta\alpha).$$

ART. **12**. The sum of a scalar and a vector is called a quaternion because it involves four independent numbers, such as the scalar and the three coefficients of the vector when resolved along three given directions (Art. 8).

Thus the product of a pair of vectors is a quaternion, and conversely, every quaternion may be expressed as a product of a pair of vectors. If q is a quaternion, if Sq is its scalar part and Vq its vector part, so that

$$q = Sq + Vq\,;$$

if α and β' are two vectors at right angles to one another and to Vq, so that $V\alpha\beta' = Vq$; and if $\beta - \beta'$ is the vector parallel to α, for which $S\alpha(\beta - \beta') = Sq$, then we have

$$Vq = V\alpha\beta \text{ because } V\alpha(\beta - \beta') = 0\,; \quad Sq = S\alpha\beta \text{ because } S\alpha\beta' = 0,$$

and therefore $$q = \alpha\beta,$$

or the quaternion has been reduced to the product of a pair of vectors.

Scalars and vectors may be regarded as simply degraded cases of quaternions.

The *sum* of any number of quaternions we define to be the sum of their scalar parts plus the sum of their vector parts. Addition of scalars is associative and commutative, and likewise addition of vectors (Art. 3). It follows that addition of quaternions is associative and commutative.

ART. **13**. We next define the product of a quaternion and a vector to be distributive with respect to the scalar and the vector of the quaternion. Thus

$$\gamma q = \gamma(Sq + Vq) = \gamma Sq + \gamma Vq, \quad q\gamma = (Sq + Vq)\gamma = Sq\,.\,\gamma + Vq\,.\,\gamma.$$

The products γVq and $Vq\,.\,\gamma$ fall under formula (B), and we define that multiplication of a scalar and a vector is commutative, so that $\gamma Sq = Sq\,.\,\gamma$.

Thus we can interpret expressions such as $\alpha\,.\,\beta\gamma$ or $\alpha\beta\,.\,\gamma$ (the product of α into the product $\beta\gamma$ and the product of the product $\alpha\beta$ into γ), and we see that they are distributive with respect to the three vectors, so that

$$\Sigma\alpha\,.\,\Sigma\beta\Sigma\gamma = \Sigma\Sigma\Sigma\alpha\,.\,\beta\gamma, \quad \Sigma\alpha\Sigma\beta\,.\,\Sigma\gamma = \Sigma\Sigma\Sigma\alpha\beta\,.\,\gamma.$$

We shall now prove that the products are associative, so that we may omit the points, and to this end we shall consider the laws of combination of three mutually rectangular unit-vectors, i, j and k.

ART. **14.** Let any three mutually rectangular unit-vectors, i, j and k, be drawn so that rotation round i from j to k is positive.

According to the usual convention, if i and j are in the plane of the paper, k will be directed vertically upwards, and it is seen at

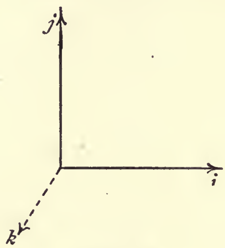

FIG. 10.

once that rotation round j from k to i, and also round k from i to j is positive (Fig. 10).

We have then, because the vectors are mutually perpendicular and of unit length,

$$Sjk = Ski = Sij = 0\,;\;\; Si^2 = Sj^2 = Sk^2 = -1\,;\; \text{..........(Art. 9)}$$

$$Vjk = i,\; Vki = j,\; Vij = k\,;\; Vkj = -i,\; Vik = -j,\; Vji = -k\,;\;\text{(Art. 10)}$$

and by formula (B) it follows at once that

$$i^2 = j^2 = k^2 = -1,\;\; jk = i = -kj,\;\; ki = j = -ik,\;\; ij = k = -ji. \ldots\text{(C)}$$

Let us now, as in the last article, form the ternary products of these vectors. We have by the relations just given

$$i\,.\,jk = i\,.\,i = -1 = k\,.\,k = ij\,.\,k = ijk,$$

$$i^2\,.\,j = -j = +i\,.\,k = i\,.\,ij = i^2 j,$$

$$i\,.\,j^2 = -i = +k\,.\,j = ij\,.\,j = ij^2,$$

the points being omitted as they are seen to be unnecessary. Similarly, for every ternary product of i, j and k, the points may be shown to be unnecessary.

For quaternary products, let ι, κ, λ, μ each denote some one of the three symbols i, j, k, then

$$\iota\,.\,\kappa\lambda\mu = \iota\,.\,\kappa\,.\,\lambda\mu = \iota\kappa\,.\,\lambda\mu = \iota\kappa\,.\,\lambda\,.\,\mu = \iota\kappa\lambda\,.\,\mu = \iota\kappa\lambda\mu,$$

because, for example, $\iota\,.\,\kappa\,.\,\lambda\mu$ is a ternary product, as $\lambda\mu$ must be $\pm i$, $\pm j$, $\pm k$ or -1. In this way all products of the symbols i, j, k are seen to be associative.

It may be a useful exercise to show that the associative law enables us to deduce all the relations (C) from Hamilton's fundamental formula (A),

$$i^2=j^2=k^2=ijk=-1. \qquad\text{(A)}$$

For example, $i\,.\,ijk=-i$ gives $jk=i$.

Ex. 1. Prove that

$$ijk=jki=kij=-1=-kji=-jik=-ikj.$$

Ex. 2. If the symbols $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ obey the laws,*

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=+1\,;\ \ \mathbf{jk}=\mathbf{i},\ \mathbf{ki}=\mathbf{j},\ \mathbf{ij}=\mathbf{k}\,;\ \ \mathbf{kj}=-\mathbf{i},\ \mathbf{ik}=-\mathbf{j},\ \mathbf{ji}=-\mathbf{k},$$

prove that their multiplication is dissociative.

[$\mathbf{i}^2.\mathbf{j}=+\mathbf{j}$ but $\mathbf{i}\,.\,\mathbf{ij}=\mathbf{i}\,.\,\mathbf{k}=-\mathbf{j}$.]

ART. **15.** We can now show that multiplication of vectors is associative. Let any three vectors, α, β and γ be expressed in terms of i, j, k, so that

$$\alpha=xi+yj+zk,\quad \beta=x'i+y'j+z'k,\quad \gamma=x''i+y''j+z''k.$$

By Art. 13,

$$\alpha\,.\,\beta\gamma=\Sigma\Sigma\Sigma xi\,.\,y'jz''k=\Sigma\Sigma\Sigma xy'z''i\,.\,jk=\Sigma\Sigma\Sigma xy'z''ijk,$$

$$\alpha\beta\,.\,\gamma=\Sigma\Sigma\Sigma\, xiy'j\,.\,z''k=\Sigma\Sigma\Sigma xy'z''ij\,.\,k=\Sigma\Sigma\Sigma xy'z''ijk,$$

so that $$\alpha\,.\,\beta\gamma=\alpha\beta\,.\,\gamma=\alpha\beta\gamma,$$

and similarly for all products of higher orders.

Hence multiplication of quaternions is associative, for a quaternion may be expressed as the product of a pair of vectors.

It now appears (compare Art. 13) that the product of any number of vectors taken in any given order is a definite quaternion.

ART. **16.** The division of vectors may be reduced to multiplication. By formula (B) the square of a vector is

$$\alpha^2=\mathrm{S}\,.\,\alpha^2=-(\mathrm{T}\alpha)^2;\ \text{ so that }\ \alpha\,.\,\frac{-\alpha}{(\mathrm{T}\alpha)^2}=1,$$

and thus it appears that $-\alpha:(\mathrm{T}\alpha)^2$ is the reciprocal of the vector α, say α^{-1} or $\frac{1}{\alpha}$. The vector α^{-1} is opposite to α in direction,

* Mr. Oliver Heaviside bases his vectorial Algebra on these laws. Prof. Knott (*Recent Innovations in Vector Theory*, Proc. R.S.E., 1892-3) draws attention to papers written by the Rev. M. O'Brien in the years 1846-52, in which the square of a vector is taken to be positive.

and its tensor is the reciprocal of that of α. We can therefore interpret products such as

$$\beta\alpha^{-1} \text{ and } \alpha^{-1}\beta,$$

and the first of these we shall call the quotient of β by α, and denote it by

$$\frac{\beta}{\alpha} \text{ or } \beta:\alpha.$$

The reciprocal of any product of vectors is the product of their reciprocals taken in the reverse order. For if

$$Q=\alpha\beta\gamma\delta, \quad Q'=\delta^{-1}\gamma^{-1}\beta^{-1}\alpha^{-1},$$

we have

$$QQ'=1$$

in virtue of the associative law. Similarly, the reciprocal of a product of quaternions is the product of the quaternions taken in the reverse order. Hence every quotient of vectors or of quaternions is a quaternion; and more generally every combination of quaternions by the processes of addition, subtraction, multiplication and division is a quaternion.

Ex. 1. Prove that

$$\mathrm{S}\frac{\gamma}{\alpha+\beta}=-\frac{\mathrm{S}\gamma\alpha}{\mathrm{T}(\alpha+\beta)^2} \text{ if } \mathrm{S}\gamma\beta=0.$$

Ex. 2. Distinguish between the expressions

$$\frac{\delta}{\gamma}\cdot\frac{\beta}{\alpha} \text{ and } \frac{\delta\beta}{\gamma\alpha}.$$

[These may be written $\delta\gamma^{-1}\beta\alpha^{-1}$ and $\delta\beta\alpha^{-1}\gamma^{-1}$.]

Ex. 3. Prove that

$$\frac{\beta\gamma}{\alpha\gamma}=\frac{\beta}{\alpha}; \quad \frac{\gamma\beta}{\gamma\alpha}=\gamma\left(\frac{\beta}{\alpha}\right)\gamma^{-1}.$$

ART. **17.** The *conjugate* Kq of a quaternion q is defined by the relation

$$\mathrm{K}q=\mathrm{S}q-\mathrm{V}q.$$

for ctors β → If then $q=\alpha\beta$, we have $\mathrm{K}q=\beta\alpha$ (Art. 11), and

$$q\mathrm{K}q=\alpha\beta\beta\alpha=\alpha^2\beta^2=\mathrm{K}qq=\mathrm{T}\alpha^2\mathrm{T}\beta^2 \text{ (Art. 16).}$$

The products of the tensors of the vectors into which a quaternion is resolvable is therefore independent of any particular selection of the vectors since Sq and Vq are independent of any particular pair of vectors; and the square of this product is

$$q\mathrm{K}q=(\mathrm{S}q+\mathrm{V}q)(\mathrm{S}q-\mathrm{V}q)=(\mathrm{S}q)^2-(\mathrm{V}q)^2=\mathrm{K}qq=(\mathrm{T}q)^2,$$

if we call this constant product of tensors, the *tensor* (Tq) of the quaternion.

Again,

$$q = \alpha\beta = \mathrm{T}\alpha\,.\,\mathrm{U}\alpha\,.\,\mathrm{T}\beta\,.\,\mathrm{U}\beta = \mathrm{T}\alpha\mathrm{T}\beta\,.\,\mathrm{U}\alpha\mathrm{U}\beta = \mathrm{T}q\,.\,\mathrm{U}q$$

and $\mathrm{U}q = \mathrm{U}\alpha\mathrm{U}\beta$ is called the *versor* of the quaternion. If $\pi - \angle q$ is the angle between the vectors α and β, which is less than two right angles and measured from α to β, we see by the definitions of $\mathrm{S}q$ and $\mathrm{V}q$ that (Arts. 9 and 10)

$$\mathrm{S}q = \mathrm{T}q\cos\angle q, \quad \mathrm{TV}q = \mathrm{T}q\sin\angle q.$$

The angle $\angle q$ is called the *angle* of the quaternion, and is independent of any particular set of vectors α, β.

A plane at right angles to $\mathrm{V}q$ is called the *plane* of the quaternion and $\mathrm{UV}q$ is called the *axis*.

Ex. 1. Prove that

$$\begin{aligned}\mathrm{K}q &= w - ix - jy - kz,\\ \mathrm{T}q &= \sqrt{(w^2+x^2+y^2+z^2)},\\ \mathrm{TV}q &= \sqrt{(x^2+y^2+z^2)},\\ \mathrm{UV}q &= (+ix+jy+kz) : \sqrt{(+x^2+y^2+z^2)},\\ \mathrm{U}q &= (w+ix+jy+kz) : \sqrt{(w^2+x^2+y^2+z^2)},\\ \mathrm{TVU}q &= \sqrt{\frac{x^2+y^2+z^2}{w^2+x^2+y^2+z^2}},\end{aligned}$$

where $$q = w + ix + jy + kz.$$

Ex. 2. Write down the analogous functions of $\mathrm{K}q$ in terms of x, y, z and w.

Ex. 3. Prove that $\alpha^{-1}\beta = \mathrm{K}\,.\,\beta\alpha^{-1}$.

Ex. 4. What is the nature of q if $q = \mathrm{K}q$? If $q = -\mathrm{K}q$?

ART. **18.** We can always reduce a quaternion to a quotient of vectors (Arts. 12, 16), and write

$$q = \frac{\beta}{\alpha} = \frac{\mathrm{OB}}{\mathrm{OA}}, \quad \mathrm{T}q = \frac{\mathrm{TOB}}{\mathrm{TOA}}, \quad \mathrm{U}q = \frac{\mathrm{UOB}}{\mathrm{UOA}}, \quad \mathrm{S}q = \frac{\mathrm{OA}'}{\mathrm{OA}}, \quad \mathrm{V}q = \frac{\mathrm{A}'\mathrm{B}}{\mathrm{OA}}, \quad \angle q = \mathrm{AOB},$$

the line BA′ being drawn perpendicular to OA.

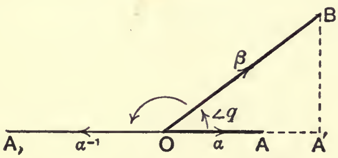

FIG. 11.

Thus the shape of the triangle AOB is constant for a given quaternion. From this point of view, a quaternion is called by Hamilton a ratio of vectors, as it depends on their relative magnitudes and on their relative directions.

It is not difficult to show that the conjugate (see Fig. 12)

$$\mathrm{K}q=\frac{\mathrm{OB}'}{\mathrm{OA}} \text{ if } q=\frac{\mathrm{OB}}{\mathrm{OA}},$$

for $$q+\mathrm{K}q=2\mathrm{S}q, \quad q-\mathrm{K}q=2\mathrm{V}q.$$

The triangle AOB′ is inversely similar* to AOB.

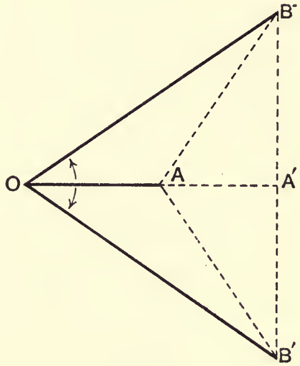

FIG. 12.

ART. **19**. Conversely, if the product $q\alpha$ is a vector β, it is evident that α and β are both at right angles to $\mathrm{V}q$. And if α is any vector at right angles to $\mathrm{V}q$, $q\alpha$ is a vector making a constant angle ($\angle q$) with α, and having its length $\mathrm{T}q$ times that of α. In other words, regarding the quaternion as an operator, it turns vectors *in its plane* through a given angle, and alters their lengths in a given ratio. In particular we may regard a vector as turning vectors at right angles to it through a right angle, and altering their lengths proportionately to its own.

The versor $\mathrm{U}q$ turns vectors in its plane through the angle $\angle q$ but leaves their lengths unaltered. The tensor $\mathrm{T}q$ alters the lengths of all vectors in a given ratio. The total effect produced by q on a vector in its plane may be considered to be effected in two stages or at once as indicated by the relation

$$\beta=q\alpha=\mathrm{T}q\,.\,\mathrm{U}q\,.\,\alpha=\mathrm{U}q\,.\,\mathrm{T}q\,.\,\alpha.$$

ART. **20**. The results of articles 18, 15 and 16 afford an extremely elegant construction for the product of two quaternions q and r. Take any vector OB along the line of intersection of the planes of the two quaternions. Make the triangle BOC in

* Hamilton uses the phrases *direct similitude* and *inverse similitude* in the sense that two directly similar figures in a plane appear to have the same shape; while of two inversely similar figures one has the same shape as the reflection of the other in a mirror.

the plane of r similar to the triangle determined by r (Art. 18); make AOB in the plane of q similar to the triangle of q; then, by the associative principle (Fig. 13)

$$rq=\frac{\mathrm{OC}}{\mathrm{OA}}\left(=\frac{\mathrm{OC}}{\mathrm{OB}}\cdot\frac{\mathrm{OB}}{\mathrm{OA}}=\mathrm{OC}\,.\,\mathrm{OB}^{-1}\,.\,\mathrm{OB}\,.\,\mathrm{OA}^{-1}\right).$$

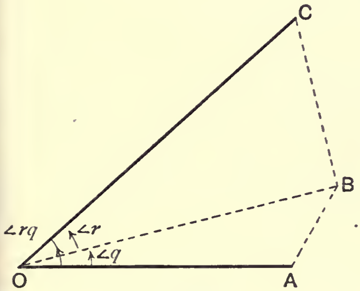

FIG. 13.

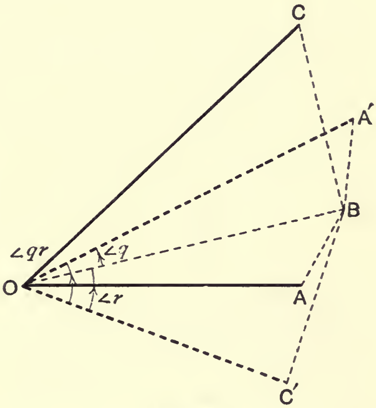

FIG. 14.

If the triangles BOA′ and C′OB are respectively coplanar with and similar to AOB and BOC, the second product is (Fig. 14)

$$qr=\frac{\mathrm{OA}'}{\mathrm{OC}'}\left(=\frac{\mathrm{OA}'}{\mathrm{OB}}\cdot\frac{\mathrm{OB}}{\mathrm{OC}'}\right).$$

Ex. 1. Prove that $\mathrm{K}(rq)=\mathrm{K}q\mathrm{K}r$.

[Take $\mathrm{C}_{,}$ on OC and $\mathrm{A}_{,}$ on OA so that $\mathrm{C}_{,}\mathrm{OB}$ and $\mathrm{BOA}_{,}$ are inversely similar to BOC and AOB, and the triangle $\mathrm{A}_{,}\mathrm{OC}_{,}$ is inversely similar to COA. Art. 18.]

Ex. 2. The product of the conjugates of any number of quaternions is the conjugate of their product in reverse order.
[By Ex. 1, $\mathrm{K}(p\,.\,qr)=\mathrm{K}(qr)\,.\,\mathrm{K}p$, etc.]

Ex. 3. Show that

$$\mathrm{S}p_1p_2p_3\ldots p_n=\tfrac{1}{2}[p_1p_2\ldots p_n+\mathrm{K}p_n\mathrm{K}p_{n-1}\ldots\mathrm{K}p_1],$$
$$\mathrm{V}p_1p_2p_3\ldots p_n=\tfrac{1}{2}[p_1p_2\ldots p_n-\mathrm{K}p_n\mathrm{K}p_{n-1}\ldots\mathrm{K}p_1].$$

Ex. 4. If $\alpha_1\alpha_2\ldots\alpha_n$ are n vectors, and if $\Pi\alpha=\alpha_1\alpha_2\ldots\alpha_n$, $\Pi'\alpha=\alpha_n\alpha_{n-1}\ldots\alpha_1$, show that

$$\mathrm{S}\Pi\alpha=\tfrac{1}{2}\Pi\alpha+\tfrac{1}{2}(-)^n\Pi'\alpha,$$
$$\mathrm{V}\Pi\alpha=\tfrac{1}{2}\Pi\alpha-\tfrac{1}{2}(-)^n\Pi'\alpha.$$

Ex. 5. Prove that

$$\mathrm{S}pq=\mathrm{S}qp\,;\quad \mathrm{TV}pq=\mathrm{TV}qp\,;\quad \angle pq=\angle qp.$$

Ex. 6. Prove that $p\mathrm{K}q+q\mathrm{K}p=2\mathrm{S}\,.\,p\mathrm{K}q=2\mathrm{S}\,.\,q\mathrm{K}p$.

Ex. 7. Prove that the tensor of a product of any number of quaternions is independent of their order.

Ex. 8. Prove that the versor of a product of any number of quaternions is the product of the versors taken in the same order.

Ex. 9. Show that three quaternions cannot in general be reduced simultaneously to the forms

$$p=\frac{\delta}{\gamma},\ q=\frac{\gamma}{\beta},\ r=\frac{\beta}{\alpha}.$$

Ex. 10. Prove that the scalar of a product of any number of quaternions is unchanged when the quaternions are cyclically transposed.

Ex. 11. Prove that the tensor of the vector part of a product of quaternions remains unchanged for cyclical transposition.

Ex. 12. Prove the identity

$$\begin{aligned}(ww'-xx'-yy'-zz')^2+(wx'+w'x+yz'-y'z)^2 \\ +(wy'+w'y+zx'-z'x)^2+(wz'+w'z+xy'-x'y)^2 \\ =(w^2+x^2+y^2+z^2)(w'^2+x'^2+y'^2+z'^2).\end{aligned}$$

[See Ex. 1 of this series and Ex. 1, Art. 17. This identity is of historical interest as regards the discovery of quaternions. *See* Graves's *Life of Sir William Rowan Hamilton*, vol. ii., p. 437.]

ART. **21.** The multiplication of versors, to which the multiplication of quaternions may be reduced by separating the tensors, admits of a simple spherical representation.

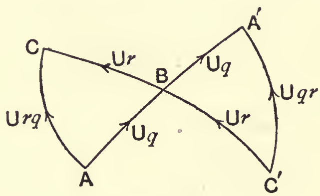

FIG. 15.

A versor is represented by a directed great circle arc belonging to a definite great circle (the plane of the versor) and having a definite length (the angle of the versor). From the figure (Fig. 15)

$$\mathrm{U}rq=\frac{\mathrm{OC}}{\mathrm{OA}}=\frac{\mathrm{OC}}{\mathrm{OB}}\cdot\frac{\mathrm{OB}}{\mathrm{OA}}=\mathrm{U}r\mathrm{U}q\,;$$

$$\mathrm{U}qr=\frac{\mathrm{OA}'}{\mathrm{OC}'}=\frac{\mathrm{OA}'}{\mathrm{OB}}\cdot\frac{\mathrm{OB}}{\mathrm{OC}'}=\mathrm{U}q\mathrm{U}r.$$

The spherical triangles ABC and A′BC′ are inversely equal.

The construction recalls the construction for the sum of vectors, and it is allowable to write

$$\frown{\mathrm{AC}}=\frown{\mathrm{BC}}+\frown{\mathrm{AB}}\,;\quad \frown{\mathrm{C'A'}}=\frown{\mathrm{BA'}}+\frown{\mathrm{C'B}}=\frown{\mathrm{AB}}+\frown{\mathrm{BC}}.$$

This addition of vector-arcs is not commutative, for $\widehat{\text{C}'\text{A}'}$ is not generally equal to $\widehat{\text{AC}}$—equality of these vector-arcs requiring equality of length, similarity of direction and coplanarity.

Two quaternions are commutative in order of multiplication if, and only if, they are coplanar. A necessary condition for commutation is that the arcs AC and A′C′ should belong to the same great circle. If OB is not coplanar with this circle, B must be its pole. In this case the angles of the versors are right, and

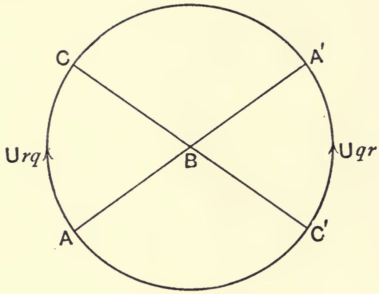

FIG. 16.

the versors are unit vectors. But a glance at the figure shows that the versor products have oppositely directed angles, and the products are therefore unequal (compare figs. 15 and 16).

For coplanar versors, the arc AB = CD in fig. 17, and

$$\mathrm{U}r\mathrm{U}q=\frac{\mathrm{OC}}{\mathrm{OB}}=\frac{\mathrm{OB}}{\mathrm{OA}}\cdot\frac{\mathrm{OC}}{\mathrm{OA}}=\frac{\mathrm{OD}}{\mathrm{OB}}=\frac{\mathrm{OD}}{\mathrm{OC}}\cdot\frac{\mathrm{OC}}{\mathrm{OB}}=\mathrm{U}q\mathrm{U}r.$$

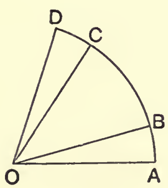

FIG. 17.

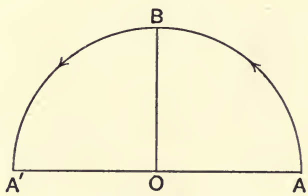

FIG. 18.

That the square of a right versor is equal to negative unity is well illustrated by fig. 18, for which

$$\left(\frac{\mathrm{OB}}{\mathrm{OA}}\right)^2=\frac{\mathrm{OA}'}{\mathrm{OB}}\cdot\frac{\mathrm{OB}}{\mathrm{OA}}=\frac{\mathrm{OA}'}{\mathrm{OA}}=-1,$$

the vector OB being perpendicular to A′A.

Replacing Urq in fig. 15 by Up, we have the new figure (fig. 19), since Ur=Upq^{-1} and Uqr=Uqpq^{-1}. The point Q is the pole of the versor Uq or the extremity of the vector UVq.

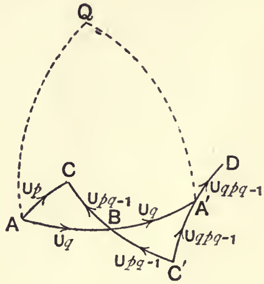

FIG. 19.

The arcs AC and A′D are equal, and equally inclined to the great circle ABA′ since the angles of the triangles ABC and A′BC′ are equal. Thus AC may be changed into A′D by a rotation round Q through the angle AQA′, double the angle of the quaternion q. The vector UVp to the pole of the arc AC is transformed into UVqpq^{-1} by the same rotation. Now V$qpq^{-1}=q\,.\,$V$p\,.\,q^{-1}$ because V$(q\,.\,$S$p\,.\,q^{-1})=0$, S$(q\,.\,$V$p\,.\,q^{-1})=0$, and accordingly a conical rotation round the axis of q and through double its angle changes an arbitrary vector ρ into the vector $q\rho q^{-1}$.

Ex. 1. If OP is the vector from a fixed point to a point in a rigid body, rotation of the body round an axis OQ=Vq through an angle $2\angle q$ carries the point P to P′, where OP′$=q\,.\,$OP$\,.\,q^{-1}$.

Ex. 2. The displacement produced by the rotation is

$$\text{PP}'=q\,.\,\text{OP}\,.\,q^{-1}-\text{OP}.$$

Ex. 3. A translation of the body carries a point from P to P″, where PP″$=\delta$ is the same for all points of the body.

Ex. 4. If the body is first rotated, as in Example 1, and then translated the displacement of P is

$$\delta+q\,.\,\text{OP}\,.\,q^{-1}-\text{OP}\,;$$

while if it is first translated and then rotated, the displacement is

$$q(\delta+\text{OP})q^{-1}-\text{OP}.$$

Ex. 5. If the body is first rotated about one axis OQ and then about another OR,

$$\text{OP}'=rq\,.\,\text{OP}\,.\,q^{-1}r^{-1}=rq\,.\,\text{OP}\,.\,(rq)^{-1}.$$

Ex. 6. If the first rotation is now reversed, the position of the point P′ is P″, where

$$\text{OP}''=q^{-1}rq\,.\,\text{OP}\,.\,q^{-1}r^{-1}q.$$

Ex. 7. A body receives rotations about two intersecting axes. Prove that the order in which these rotations are effected is of importance.

[The displacements of a point are

$$qr\,.\,\text{OP}\,.\,r^{-1}q^{-1}-\text{OP} \quad\text{and}\quad rq\,.\,\text{OP}\,.\,q^{-1}r^{-1}-\text{OP},$$

and these are generally different unless $qr=rq$, but then the quaternions are coplanar and the rotations take place about one and the same axis.]

Ex. 8. Find the reflection of a body in a plane mirror.

[The point O being on the mirror, which is perpendicular to λ, the vector $\lambda \,.\, \text{OP} \,.\, \lambda^{-1}$ is the result of rotating OP through two right angles round the normal. Reversing the direction of this vector, the vector to the image of the point P is $\text{OP}' = -\lambda \,.\, \text{OP} \,.\, \lambda^{-1}$.]

Ex. 9. Successive reflection in two mirrors is equivalent to a rotation round the line of intersection of the mirrors through double the angle between the mirrors.

[Here $-\mu(-\lambda \,.\, \text{OP} \,.\, \lambda^{-1})\mu^{-1} = +\mu\lambda \,.\, \text{OP} \,.\, \lambda^{-1}\mu^{-1}$. Also $\angle\mu\lambda = \theta$, where θ is the angle between the mirrors, and $2\angle\mu\lambda = 2\theta$.]

Ex. 10. Given three lines intersecting in a point, it is required to draw three planes, each through one of the lines, so that the lines of intersection in one plane may be equally inclined to the contained line.

When is the problem indeterminate?

[Let α, β, γ be the vectors of the given lines. The sought lines of intersection are $\mathrm{V}\beta\alpha\gamma$, $\mathrm{V}\gamma\beta\alpha$, $\mathrm{V}\alpha\gamma\beta$. Compare Art. 31, p. 31.]

ART. **22.** The laws of combination of the five symbols

$$\mathrm{S},\ \mathrm{V},\ \mathrm{K},\ \mathrm{T} \text{ and } \mathrm{U}$$

may be summarized in the symbolical multiplication table:

	S	V	K	T	U
S	S	0	S	T	SU
V	0	V	−V	0	VU
K	S	−V	1	T	KU
T	±S	TV	T	T	—
U	—	UV	UK	—	U

to be read from the left. For example, the tensor of the vector of a quaternion is $\mathrm{TV}q$; the scalar of the vector is 0; the tensor of the scalar is $\pm\mathrm{S}q$ according as $\mathrm{S}q$ is positive or negative. A positive scalar may be regarded as the quotient of two vectors having the same direction; for a negative scalar the directions are opposite. Hence we may write $\mathrm{US}q = \pm 1$ according as $\mathrm{S}q$ is positive or negative. The versor of a zero quaternion must be regarded as arbitrary, unless we know a law according to which the quaternion diminished indefinitely. $\mathrm{TU}q = 1 = \mathrm{UT}q$ for all quaternions. The versor of the conjugate and the conjugate of the versor of a quaternion are easily seen to be equal to one another and to the reciprocal of the versor. The symbols T and U are not distributive like the symbols S, K and V. Apart from change of sign, it is easy to see that the only new combination arising from further repetition of the symbols is $\mathrm{TVU}q$ ($= \sin \angle q$).

It is necessary to make some convention concerning the notation to be employed when we wish to denote for example the square of the scalar of a quaternion q or the scalar of the square of the quaternion. There can be no mistake if we employ brackets and write $(\mathrm{S}q)^2$ for the square of the scalar and $\mathrm{S}(q^2)$ for the scalar of the square, and whenever there is the least fear of confusion brackets should be used. One of the great advantages of quaternions is the extreme brevity of the notation. Another and still greater advantage is its great explicitness, and this should never be sacrificed for the sake of a few brackets. Hamilton writes $\mathrm{S} \,.\, q^2$ for the scalar of the square and $\mathrm{S}q^2$ for the square of the scalar whenever there is no fear of confusion, and he uses the notation $\mathrm{V} \,.\, q^2$ and $\mathrm{V}q^2$ in a similar sense and in conformity with the established notation $\mathrm{d} \,.\, x^2$ and $\mathrm{d}x^2$ for the differential of x^2 and for the square of the differential of x. Some eminent authorities, Tait for instance, in conformity with the notation $\cos^2 x = (\cos x)^2$, write $\mathrm{S}^2 q$ instead of $\mathrm{S}q^2$, though in strictness this would mean $\mathrm{S} \,.\, \mathrm{S}q$ $(=\mathrm{S}q)$. But considering the enormous care Hamilton took with his notation we prefer to abide by his convention. No confusion can arise with respect to $\mathrm{T} \,.\, q^2$ or $\mathrm{T}q^2$ or $(\mathrm{T}q)^2$, for the tensor of the square is the square of the tensor, and similarly $\mathrm{U} \,.\, q^2 = \mathrm{U}q^2 = (\mathrm{U}q)^2$ and $\mathrm{K} \,.\, q^2 = (\mathrm{K}q)^2 = \mathrm{K}q^2$. The expression $\mathrm{S}p \,.\, q$ means the product of $\mathrm{S}p$ into q, and it is well when possible to write this in the equivalent form $q\mathrm{S}p$, while $\mathrm{S} \,.\, pq$ is the scalar of the product pq, but if the expressions are at all complicated, it is safer to write $(\mathrm{S}p)q$ and $\mathrm{S}(pq)$.

An imaginary quaternion

$$Q = p + \sqrt{-1} \,.\, q,$$

where p and q are real quaternions and where $\sqrt{-1}$ is the imaginary symbol of algebra regarded as a scalar commutative with all quaternions, is called a *biquaternion* by Hamilton. Similarly he calls imaginary vectors $(\alpha + \sqrt{-1} \,.\, \beta)$ *bivectors* and imaginary scalars, *biscalars*. No ambiguity attaches to

$$\mathrm{S}Q = \mathrm{S}p + \sqrt{-1}\mathrm{S}q, \text{ or to } \mathrm{V}Q = \mathrm{V}p + \sqrt{-1}\mathrm{V}q,$$

and the only ambiguity in $\mathrm{T}Q$ is one of sign, and this Hamilton removes as follows. He writes

$$\mathrm{T}Q = x + \sqrt{-1} \,.\, y,$$

where x and y are real scalars and where x is positive, and in order to determine x and y he employs the relation (Art. 17)

$$(\mathrm{T}Q)^2 = Q\mathrm{K}Q = p\mathrm{K}p - q\mathrm{K}q + \sqrt{-1}(p\mathrm{K}q + q\mathrm{K}p),$$

or

$$(\mathrm{T}Q)^2 = \mathrm{T}p^2 - \mathrm{T}q^2 + 2\sqrt{-1}\mathrm{S} \,.\, p\mathrm{K}q = x^2 - y^2 + 2\sqrt{-1} \,.\, xy,$$

observing that $q\mathrm{K}p = \mathrm{K} \,.\, p\mathrm{K}q$, so that the imaginary part of $(\mathrm{T}Q)^2$ may be written

$$2\sqrt{-1}\mathrm{S} \,.\, p\mathrm{K}q, \text{ or } 2\sqrt{-1}\mathrm{S} \,.\, q\mathrm{K}p.$$

Equating reals and imaginaries we find, from

$$x^2 - y^2 = \mathrm{T}p^2 - \mathrm{T}q^2 \text{ and } xy = \mathrm{S} \,.\, p\mathrm{K}q,$$

that the real positive value of x is

$$x = \{\tfrac{1}{2}(\mathrm{T}p^2 - \mathrm{T}q^2) + [\tfrac{1}{4}(\mathrm{T}p^2 - \mathrm{T}q^2)^2 + (\mathrm{S} \,.\, p\mathrm{K}q)^2]^{\frac{1}{2}}\}^{\frac{1}{2}}.$$

It may happen that $T(QQ')$ is $-TQTQ'$ instead of $+TQTQ'$ where Q and Q' are biquaternions. In other particulars ambiguity does not arise.

The tensor of a biquaternion may vanish, and in this case we have an equation such as

$$QQ'=0,$$

where $Q'=KQ$ without having either Q or Q' zero. The conditions are

$$Tp^2=Tq^2 \text{ and } S.pKq=0,$$

and when these are satisfied, the biquaternion Q is called by Hamilton a *nullifier*. A few examples will be found in Chap. IV.; and the *Lectures on Quaternions* (Arts. 669-675), from which this account of biquaternions has been taken, may be consulted with advantage.*

Ex. 1. Prove that combinations of the symbols prefixed to q lead to one or other of the following :

$$Sq,\ Vq,\ Kq,\ Tq,\ Uq;\ \ TVq,\ SUq,\ VUq,\ TVUq,\ (Uq)^{-1},\ UVq.$$

Ex. 2. Express these functions in terms of x, y, z, w, i, j and k. (See Ex. 1, Art. 17, p. 13.)

Ex. 3. Express these functions in terms of the tensor, axis and angle of the quaternion.

Ex. 4. Show that the vectors $UVpq$ and $UV.UpUq$ are identical.

Ex. 5. If α, β and γ are vectors, prove that V is a redundant symbol in $S.\alpha V.\beta\gamma$.

Ex. 6. Find the difference of the expressions $S.pqr$ and $S.pV.qr$.

Ex. 7. If $UVp=VUp$, prove that $Sp=0$.

Ex. 8. What inference can be drawn from the equation $Vq=VUq$? and what from $Vq=Uq$?

Ex. 9. Prove that

$$T(\gamma+\beta)>\pm(T\gamma-T\beta) \text{ unless } U\gamma=-U\beta,$$

and find the relation in the exceptional case.

Ex. 10. Show that

$$Tq+Tp>T(q+p) \text{ unless } q=xp,\ x>0.$$

Ex. 11. Show that

$$Tq+Sq>0 \text{ unless } \angle q=\pi.$$

EXAMPLES TO CHAPTER II.

Ex. 1. Prove that $V(\alpha-\beta)(\alpha+\beta)=2V\alpha\beta$ and assign the geometrical interpretation.

Ex. 2. Show similarly that $S(\alpha-\beta)(\alpha+\beta)=\alpha^2-\beta^2$ and interpret.

Ex. 3. Under what conditions is $(\alpha+\beta)(\alpha-\beta)$ equal to $\alpha^2-\beta^2$?

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*Clifford uses the word biquaternion in another sense, and Prof. A. M'Aulay has rechristened Clifford's biquaternions, and has written a large book entitled "Octonions: a Development of Clifford's Biquaternions." (Cambridge, 1898.) It does not seem to be unreasonable to retain Hamilton's convenient word for the purpose for which it was coined.

Ex. 4. Establish the identity connecting three quaternions,

$$p^3+q^3+r^3=pqr+qrp+rpq, \quad \text{where} \quad p+q+r=0.$$

Ex. 5. If the relation

$$\frac{1}{\beta}-\frac{1}{\alpha}=\frac{\alpha-\beta}{\alpha\beta}$$

connects two vectors α and β, prove that $\alpha\beta^{-1}\alpha^{-1}=\beta^{-1}$ and show that the vectors are parallel.

Ex. 6. Reduce any two quaternions p and q to quotients of vectors having a common denominator, or in other words, find three vectors α, β and γ, so that

$$p=\frac{\beta}{\alpha}, \quad q=\frac{\gamma}{\alpha}.$$

Ex. 7. Prove that the relations

$$p+q=\frac{\beta+\gamma}{\alpha}, \quad p-q=\frac{\beta-\gamma}{\alpha}, \quad \text{where} \quad p=\frac{\beta}{\alpha}, \quad q=\frac{\gamma}{\alpha},$$

are consistent with the definition that the sum of quaternions is the sum of their scalar parts plus the sum of their vector parts.

Ex. 8. For any two quaternions

$$q(q^{-1}\pm r^{-1})=(r\pm q)r^{-1}; \quad q(q\pm r)^{-1}r=(r^{-1}\pm q^{-1})^{-1}.$$

Ex. 9. The sign V is superfluous in $\mathrm{S}\,.\,\alpha\mathrm{V}\beta\gamma$. Is it superfluous in

$$\mathrm{S}\,.\,\frac{\alpha}{\mathrm{V}\beta\gamma}?$$

Ex. 10. The second vector α may be omitted from $\mathrm{V}\alpha(\alpha+\beta)$. May it be omitted in $\mathrm{V}\alpha^{-1}(\alpha+\beta)$ or in $\mathrm{V}\alpha(\alpha+\beta)^{-1}$?

Ex. 11. Contrast, where necessary, the four expressions,

$$\mathrm{S}\frac{\mathrm{V}\alpha\beta}{\gamma\delta}, \quad \mathrm{S}\frac{\alpha\beta}{\mathrm{V}\gamma\delta}, \quad \mathrm{S}\frac{\mathrm{V}\alpha\beta}{\mathrm{V}\gamma\delta}, \quad \mathrm{S}\frac{\alpha\beta}{\gamma\delta}.$$

Ex. 12. The laws of refraction of light from a medium of index n into one of index n' are comprised in the relation

$$n\mathrm{V}\nu\alpha=n'\mathrm{V}\nu\alpha',$$

where ν, α and α' are unit vectors along the normal, the incident and the refracted ray, respectively.

(a) From this relation,

$$n'\alpha'=\nu\sqrt{(n'^2+n^2\mathrm{V}\nu\alpha^2)}-n\nu\mathrm{V}\nu\alpha.$$

Ex. 13. It is required to find a quaternion q and vectors α, β and γ, so that if a, b and c are three given quaternions,

$$aq=\alpha, \quad bq=\beta, \quad cq=\gamma.$$

(a) Show that

$$\frac{a}{b}=\frac{\alpha}{\beta}, \quad \frac{b}{c}=\frac{\beta}{\gamma}, \quad \frac{c}{a}=\frac{\gamma}{\alpha};$$

and explain how α, β and γ can be found from these relations; the tensor of one vector (α) being assumed. (Robert Russell.)

CHAPTER III.

FORMULAE AND INTERPRETATIONS DEPENDING ON PRODUCTS OF VECTORS.

ART. **23.** It is often useful to consider a vector as representing a directed area. Assuming any two vectors α, β, so that $V\alpha\beta$ may equal a given vector γ, we may regard γ as representing the directed area of the parallelogram determined by α and β—there being as many units of area in the parallelogram as there are units of length in γ. The shape of the area represented by a vector is arbitrary as well as its position; its magnitude and *aspect* are determinate. For there is obviously no reason why this representation should be confined to the areas of *parallelograms.*

Ex. A force is represented in magnitude and line of action by the line AB. The moment of the force at the point O is represented by

$$V \,.\, OA \,.\, AB.$$

ART. **24.** *The scalar of the product of three vectors is the volume of the parallelepiped having conterminous edges equal to the vectors.*

The transformation

$$S \,.\, \alpha\beta\gamma = S \,.\, \alpha(V\beta\gamma + S\beta\gamma) = S \,.\, \alpha V\beta\gamma$$

shows that this scalar is equal to the scalar of the binary product of α into $V\beta\gamma$—that is, it is the negative product of the projection of α on the normal $UV\beta\gamma$ to one face into the area of that face. If rotation round α from β towards γ is positive, the volume is $-S\alpha\beta\gamma$, for the angle between α and $UV\beta\gamma$ is then acute, and $S\alpha UV\beta\gamma$ is negative.

Ex. 1. If $S\alpha\beta\gamma = 0$ the vectors are coplanar, and conversely.

Ex. 2. Prove that interchange of any two vectors changes the sign of $S\alpha\beta\gamma$.

Ex. 3. Prove that

$$S\alpha\beta\gamma = S\alpha\alpha'\alpha'' \text{ if } \beta = x\alpha + \alpha', \ \gamma = y\alpha + z\alpha' + \alpha''.$$

Ex. 4. Prove the identity,

$$S(\delta-\alpha)(\delta-\beta)(\delta-\gamma)=S\beta\gamma\delta-S\alpha\gamma\delta+S\alpha\beta\delta-S\alpha\beta\gamma.$$

Ex. 5. Prove that $\pm S\text{AB}\,.\,\text{AC}\,.\,\text{AD}$ is six times the volume of the tetrahedron ABCD.

ART. **25.** The formula

$$V\,.\,\alpha V\beta\gamma=\gamma S\alpha\beta-\beta S\gamma\alpha \quad\text{.....................(I.)}$$

is very important owing to its frequent occurrence. Since the vector on the left is perpendicular to $V\beta\gamma$ it must be coplanar with β and γ—that is, it must be of the form $x\beta+y\gamma$ where x and y are scalars. But the vector is also perpendicular to α. Therefore $S\alpha(x\beta+y\gamma)=0$, so that the ratio of x to y is determined; and the vector must be parallel to

$$w(\beta S\alpha\gamma-\gamma S\alpha\beta).$$

It remains to determine w to satisfy

$$V\,.\,\alpha V\beta\gamma=w(\beta S\alpha\gamma-\gamma S\alpha\beta).$$

Multiply by $\gamma\alpha$ and take the scalar part of the product, and we have

$$S\,.\,\gamma\alpha V\,.\,\alpha V\beta\gamma=wS\gamma\alpha\beta S\alpha\gamma=S\gamma\alpha(\alpha V\beta\gamma-S\alpha V\beta\gamma)=-S\gamma\alpha S\alpha\beta\gamma,$$

so that $w=-1$.

The proof here given is merely illustrative of a general method. Hamilton's proof is as follows. Since

$$2V\,.\,\alpha V\beta\gamma=\alpha V\beta\gamma-V\beta\gamma\,.\,\alpha=\alpha(\beta\gamma-S\beta\gamma)-(\beta\gamma-S\beta\gamma)\alpha$$
$$=\alpha\beta\gamma-\beta\gamma\alpha\,;$$

on adding the pair of cancelling terms $\beta\alpha\gamma-\beta\alpha\gamma$, we have

$$2V\,.\,\alpha V\beta\gamma=(\alpha\beta+\beta\alpha)\gamma-\beta(\gamma\alpha+\alpha\gamma)=2\gamma S\alpha\beta-2\beta S\alpha\gamma.$$

Adding $\alpha S\beta\gamma$ to each side of the formula, we find the relation

$$V\,.\,\alpha\beta\gamma=\alpha S\beta\gamma-\beta S\gamma\alpha+\gamma S\alpha\beta, \quad\text{.................(II.)}$$

which is occasionally useful.

Ex. 1. Prove that

$$V\,.\,V\alpha\beta V\gamma\delta=\alpha S\beta\gamma\delta-\beta S\alpha\gamma\delta=\delta S\alpha\beta\gamma-\gamma S\alpha\beta\delta.$$

Ex. 2. Prove that $S\,.\,V\alpha\beta V\gamma\delta=S\alpha\delta S\beta\gamma-S\alpha\gamma S\beta\delta.$

[This is $S\,.\,\alpha V\beta V\gamma\delta$.]

Ex. 3. Find the direction of the common edge of the planes parallel to α and β and to γ and δ.

[The normals to the planes are parallel to $V\alpha\beta$ and $V\gamma\delta$.]

Ex. 4. Prove that $S\,.\,V\beta\gamma V\gamma\alpha V\alpha\beta=-(S\alpha\beta\gamma)^2$.

ART. **26.** The formula

$$\rho S\alpha\beta\gamma=\alpha S\beta\gamma\rho+\beta S\gamma\alpha\rho+\gamma S\alpha\beta\rho \quad\text{..............(I.)}$$

is of great importance, as it enables us to resolve a vector along three vectors α, β and γ which are not all in the same plane. It is virtually proved in Ex. 1 of the last article.

Otherwise assume, as we may, provided $S\alpha\beta\gamma$ is not zero,

$$\rho = x\alpha + y\beta + z\gamma,$$

and operate by $S\beta\gamma$ (that is, multiply by $\beta\gamma$ and take the scalar of the product). This gives $S\beta\gamma\rho = xS\alpha\beta\gamma$.

Another valuable formula is

$$\rho S\alpha\beta\gamma = V\beta\gamma S\alpha\rho + V\gamma\alpha S\beta\rho + V\alpha\beta S\gamma\rho, \quad \text{(II.)}$$

which enables us to resolve a vector ρ into components at right angles to the planes of $\alpha\beta$, $\beta\gamma$, and of $\gamma\alpha$. Assuming

$$\rho = xV\beta\gamma + yV\gamma\alpha + zV\alpha\beta$$

and operating by $S\alpha$, $S\beta$ and $S\gamma$, the unknowns x, y and z are found.

Ex. 1. Prove that $\alpha S\beta\gamma\rho + \beta S\gamma\alpha\rho + \gamma S\alpha\beta\rho = 0$ if $S\alpha\beta\gamma = 0$.

[Here $a\alpha + b\beta + c\gamma = 0$, where a, b, c are scalars. Operate by $V\alpha$, $V\beta$ and $V\gamma$ in turn, and we find $V\beta\gamma : a = V\gamma\alpha : b = V\alpha\beta : c$.]

Ex. 2. In the same case, $V\beta\gamma S\alpha\rho + V\gamma\alpha S\beta\rho + V\alpha\beta S\gamma\rho = 0$.

Ex. 3. Eliminate ρ between the equations

$$S\alpha\rho = 1, \quad S\beta\rho = 1, \quad S\gamma\rho = 1, \quad S\delta\rho = 1.$$

Ex. 4. Eliminate the scalars x and y from the relation

$$\alpha xy + \beta x + \gamma y + \delta = 0.$$

ART. **27.** To resolve a vector along and perpendicular to a given vector, observe that

$$\rho = \lambda . \lambda^{-1}\rho = \lambda S\lambda^{-1}\rho + \lambda V\lambda^{-1}\rho. \quad \text{(I.)}$$

In case the essentials of a problem turn on two vectors α and β, put $\lambda = V\alpha\beta$, and the transformation

$$\rho = V\alpha\beta S(V\alpha\beta)^{-1}\rho + \alpha S\beta(V\alpha\beta)^{-1}\rho - \beta S\alpha(V\alpha\beta)^{-1}\rho \quad \text{(II.)}$$

will often be found useful. (Compare Art. 25.)

An expression of an analogous type is

$$\rho = \frac{S\rho\alpha\beta - \alpha S\beta\rho + \beta S\alpha\rho}{V\alpha\beta}.$$

ART. **28.** The squared tensor of $\beta - \alpha$ is

$$T(\beta - \alpha)^2 = T\beta^2 + 2S\alpha\beta + T\alpha^2 \quad \text{(I.)}$$

for $\qquad (\beta - \alpha)^2 = \beta^2 - \beta\alpha - \alpha\beta + \alpha^2.$

Hence for a plane triangle

$$a^2 + b^2 - c^2 = 2ab\cos C.$$

The identities $\quad V\alpha\beta = V\alpha(\beta - \alpha) = V\beta(\beta - \alpha)$

lead to the remaining fundamental formulae of a plane triangle,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Ex. 1. If $T(\rho-\alpha)=T(\rho+\alpha)$, prove that $S\alpha\rho=0$.

Ex. 2. The equations

$$\frac{\rho}{\alpha}=K\frac{\alpha}{\rho};\quad S\frac{\rho-\alpha}{\rho+\alpha}=0;\quad T(\alpha+a\rho)=T(a\alpha+\rho);\quad T\rho=T\alpha$$

are consequences one of another.

EXAMPLES TO CHAPTER III.

Ex. 1. If $V.q\alpha=0$, where q is a real quaternion and α a real vector, show that

$$Sq=0,\quad Vq \parallel \alpha.$$

Ex. 2. The relation $V.q\alpha=V.\alpha'q$ implies $\alpha^2=\alpha'^2$, and $S.(\alpha-\alpha')Vq=0$, where Sq does not vanish. It may be written in the form

$$(\alpha-\alpha')Sq=V.(\alpha+\alpha')Vq.$$

Ex. 3. Provided Sq is not zero, the relations $\alpha'=q\alpha q^{-1}$ and $V.q\alpha=V.\alpha'q$ are equivalent.

Ex. 4. If $\alpha'=q\alpha q^{-1}$, the quaternion q is expressible in the form

$$q=\frac{x\alpha'+y}{\alpha+\alpha'},$$

where x and y are arbitrary scalars.

Ex. 5. The same quaternion may also be written

$$q=u+v(\alpha+\alpha')+wV\alpha\alpha',$$

provided a single relation connects u, v and w. Find it.

Ex. 6. If $\alpha'=q\alpha q^{-1}$ and $\beta'=q\beta q^{-1}$, show that to a scalar factor

$$q=1+\frac{V(\alpha'-\alpha)(\beta'-\beta)}{S(\alpha'+\alpha)(\beta'-\beta)}.$$

Verify that this agrees with the expression given in the last example.

Ex. 7. If three vectors α', β', γ' are derived by a conical rotation from three others, α, β and γ, prove that it is possible to determine scalars x, y and z, so that

$$x(\alpha'-\alpha)+y(\beta'-\beta)+z(\gamma'-\gamma)=0.$$

Ex. 8. If α, β and γ are any three vectors, and if q is any quaternion, we shall have

$$S.q\alpha q^{-1}\beta\gamma+S.q\beta q^{-1}\gamma\alpha+S.q\gamma q^{-1}\alpha\beta=S.q^{-1}\alpha q\beta\gamma+S.q^{-1}\beta q\gamma\alpha+S.q^{-1}\gamma q\alpha\beta.$$

Ex. 9. If three vectors satisfy the relation

$$(\alpha\beta\gamma)^2=-\alpha^2\beta^2\gamma^2,$$

they are mutually at right angles. If they satisfy

$$(\alpha\beta\gamma)^2=+\alpha^2\beta^2\gamma^2,$$

they are coplanar.

Ex. 10. Given that $V\alpha\beta\gamma\delta=0$, prove that the four vectors are coplanar, and show that the condition is equivalent to

$$U\frac{\alpha}{\beta}=\pm U\frac{\delta}{\gamma}.$$

Interpret this result.

Ex. 11. In any product of *coplanar* vectors $\alpha_1\alpha_2\alpha_3\alpha_4\dots\alpha_n$, it is allowable to transpose among themselves in any way the vectors with even suffixes and also to transpose the vectors with odd suffixes among themselves.

CHAPTER IV.

APPLICATIONS TO PLANE AND SPHERICAL TRIGONOMETRY.

Coplanar Versors.

ART. **29.** In dealing with rotations in a plane, let ι be a unit-vector perpendicular to the plane, and let angles be measured in the sense of positive rotation round ι. If

$$Uq = \cos A + \iota \sin A, \qquad \text{(I.)}$$

the versor Uq has its angle equal to A, provided A is less than two right angles, and generally whatever magnitude the angle A may have, $\angle q = A \pm m\pi$ where m is an integer. Hamilton calls A the *amplitude* of the versor Uq, the new name being introduced to avoid any confusion as to what is meant by the angle of a versor. (Compare Art. 17, p. 13.)

It follows from the laws of multiplication of quaternions (Art. 21, p. 17) that

$$\left.\begin{array}{l} U(qr) = \cos(A+B) + \iota \sin(A+B) \\ \text{if} \quad Uq = \cos A + \iota \sin A, \quad Ur = \cos B + \iota \sin B, \end{array}\right\} \qquad \text{(II.)}$$

provided A and B are less than two right angles, and this result evidently remains true when A and B are any angles whatever. But in full, since $\iota^2 = -1$,

$$\left.\begin{array}{l} Uq \, . \, Ur = (\cos A + \iota \sin A)(\cos B + \iota \sin B) \\ \quad = \cos A \cos B - \sin A \sin B + \iota(\sin A \cos B + \cos A \sin B), \end{array}\right\} \text{(III.)}$$

and therefore on comparison with (II.), since $U(qr) = Uq \, . \, Ur$, we obtain the formulae for the expansion of $\cos(A+B)$ and of $\sin(A+B)$ on equating separately the scalar and the vector parts.

The angle of $(Uq)^n$ is n times that of Uq, provided n is an integer and $n\angle q < \pi$; and generally when n is an integer, the amplitude of $(Uq)^n$ is n times that of Uq. If the amplitude of Ur is one m^{th} that of Uq, and if the two versors are coplanar, Ur is one of the m^{th} roots of Uq; or we may write

$$Ur = (Uq)^{\frac{1}{m}}. \qquad \text{(IV.)}$$

More generally the amplitude of $(\mathrm{U}q)^{\frac{n}{m}}$ is $\frac{n}{m}$ that of $\mathrm{U}q$, and in a similar manner we can interpret the expression $(\mathrm{U}q)^x$, where x is any scalar, as a versor coplanar with $\mathrm{U}q$, and having its amplitude x times that of $\mathrm{U}q$. If A is the amplitude of $\mathrm{U}q$, we may write

$$\mathrm{U}q=\iota^{\frac{2\mathrm{A}}{\pi}}, \qquad \text{(V.)}$$

for the amplitude of $\mathrm{U}q$ is $\frac{2\mathrm{A}}{\pi}$ times a right angle, and the amplitude of ι is a right angle; and still more generally, any quaternion may be expressed as a power of a vector,

$$q=\alpha^t, \text{ where } \alpha=\mathrm{UV}q\,.\,\mathrm{T}q^{\frac{\pi}{2\angle q}},\ t=\frac{2\angle q}{\pi}. \qquad \text{(VI.)}$$

Concerning the n^{th} roots of a quaternion q which are coplanar with it, it must suffice to remark that these are n^2 in number, being the solutions of the equations,

$$\left.\begin{aligned} x^n-\frac{n\,.\,n-1}{1\,.\,2}x^{n-2}y^2+\frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{1\,.\,2\,.\,3\,.\,4}x^{n-4}y^4+\text{etc.}&=a,\\ \frac{n}{1}x^{n-1}y-\frac{n\,.\,n-1\,.\,n-2}{1\,.\,2\,.\,3}x^{n-3}y^3+\text{etc.}&=b,\end{aligned}\right\} \qquad \text{(VII.)}$$

if $\qquad q=a+\iota b$ and $\sqrt[n]{q}=x+\iota y$, since $a+\iota b=(x+\iota y)^n$;

so that in addition to the n real quaternion roots whose amplitudes are

$$\frac{1}{n}\angle q,\ \frac{1}{n}\angle q+\frac{2\pi}{n},\ \ldots\ \frac{1}{n}\angle q+\frac{2(n-1)\pi}{n}, \qquad \text{(VIII.)}$$

there are $n(n-1)$ imaginary quaternion roots corresponding to the imaginary solutions of the equations (VII.).

The exponential e^q, where q is a quaternion, is defined by the formula,

$$e^q=1+q+\frac{q^2}{1\,.\,2}+\frac{q^3}{1\,.\,2\,.\,3}+\text{etc.}; \qquad \text{(IX.)}$$

and because quaternion multiplication is not commutative,

$$e^p\,.\,e^q=\Sigma\frac{p^n}{\lfloor n}\,.\,\Sigma\frac{q^n}{\lfloor n} \text{ is not } e^{p+q}=\Sigma\frac{(p+q)^n}{\lfloor n}, \qquad \text{(X.)}$$

unless q happens to be coplanar with p. In general, however, because $\mathrm{S}q$, $\mathrm{V}q$, q and $\mathrm{K}q$ are commutative in order of multiplication,

$$e^q=e^{\mathrm{S}q}\,.\,e^{\mathrm{V}q},\quad e^{\mathrm{K}q}=e^{\mathrm{S}q}e^{-\mathrm{V}q},\quad e^qe^{\mathrm{K}q}=e^{2\mathrm{S}q}e^{\mathrm{V}q-\mathrm{V}q}=e^{2\mathrm{S}q},$$

and also by the definition of e^q it follows that

$$\mathrm{K}e^q=e^{\mathrm{K}q};\quad e^q\mathrm{K}e^q=e^{q+\mathrm{K}q}=e^{2\mathrm{S}q},$$

and thus $\qquad \mathrm{T}e^q=e^{\mathrm{S}q},\ \mathrm{U}e^q=e^{\mathrm{V}q}=\cos\mathrm{TV}q+\mathrm{UV}q\sin\mathrm{TV}q, \qquad$ (XI.)

substitution in (IX.) and separation of the scalar and vector parts affording by the known formulae for the expansion of a sine or cosine the second expression for $\mathrm{U}e^q$.

If we write $q=\log q'$, where $q'=e^q=e^{\log q'}$, ...(XII.)
we have by (XI.), $\mathrm{S}\log q'=\log \mathrm{T}q'$, $\mathrm{V}\log q'=\log \mathrm{U}q'$;
and generally if p and q are any two quaternions, we may define

$$p^q=e^{q\log p}, \qquad \text{(XIII.)}$$

but as we shall not require much, or indeed any, acquaintance with the logarithm or exponential of a quaternion in the sequel, we refer to Hamilton's *Elements of Quaternions* for further details.

Ex. 1. Prove that $\alpha+\beta\sqrt{-1}$ is a square root of zero, where $\mathrm{T}\alpha=\mathrm{T}\beta$, $\mathrm{S}\alpha\beta=0$.

[See Art. 67, Ex. 1.]

Ex. 2. Show that a product pq may be zero without having p or q equal zero.

[If pq is a scalar, q must be proportional to $\mathrm{K}p$. The squared tensor of $\sqrt{-\mathrm{T}\rho^2}+\rho$ is zero. (Art. 22, p. 21.)]

Ex. 3. Show that a quaternion q satisfies an equation of the form $q^2+2xq+y=0$ when x and y are certain scalars.

Spherical Trigonometry.

ART. **30**. If α, β and γ are three coinitial and unit vectors determining a spherical triangle ABC, the whole doctrine of the spherical triangle is contained in the relation

$$\frac{\beta}{\alpha}\cdot\frac{\alpha}{\gamma}\cdot\frac{\gamma}{\beta}=1. \qquad \text{(I.)}$$

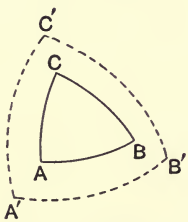

FIG. 20.

The vectors

$$\alpha'=\mathrm{UV}\frac{\gamma}{\beta}=\mathrm{UV}\beta\gamma,\ \beta'=\mathrm{UV}\gamma\alpha,\ \gamma'=\mathrm{UV}\alpha\beta,$$

terminate at the vertices of the polar triangle, rotation round these points from A to B, from B to C and from C to A being positive; and in terms of these vectors the equation may be written in the forms,

$$\frac{\beta}{\alpha}\cdot\frac{\alpha}{\gamma}=\mathrm{K}\frac{\gamma}{\beta};\ (\cos c+\gamma'\sin c)(\cos b+\beta'\sin b)=\cos a-\alpha'\sin a. \quad \text{(II.)}$$

Observing that rotation round OA from C′ to B′ is negative, the versor

$$\gamma'\beta'=\cos(\pi-\mathrm{B'C'})-\alpha\sin(\pi-\mathrm{B'C'})=\cos \mathrm{A}-\alpha\sin \mathrm{A},$$

and thus on expansion of (II.), we have

$$\cos c \cos b + \gamma' \sin c \cos b + \beta' \sin b \cos c + \sin b \sin c \cos A$$
$$- a \sin A \sin b \sin c = \cos a - a' \sin a. \quad \text{(III.)}$$

The scalar part of this equation gives the fundamental relation

$$\cos a = \cos b \cos c + \sin b \sin c \cos A; \quad \text{(IV.)}$$

while the vector part is

$$a \sin A \sin b \sin c = a' \sin a + \beta' \sin b \cos c + \gamma' \sin c \cos b. \quad \text{(V.)}$$

Operating by Sa on this vector,

$$\sin A \sin b \sin c = -\sin a S a UV\beta\gamma = -Sa\beta\gamma,$$

so that $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = -\frac{Sa\beta\gamma}{\sin a \sin b \sin c}. \quad \text{(VI.)}$$

Now (compare Art. 17 and Art. 25),

$$1 = T(a\beta\gamma)^2$$
$$= (Sa\beta\gamma)^2 - (Va\beta\gamma)^2 = (Sa\beta\gamma)^2 - (aS\beta\gamma - \beta S\gamma a + \gamma S a\beta)^2$$
$$= (Sa\beta\gamma)^2 - (a^2 S\beta\gamma^2 + \beta^2 S\gamma a^2 + \gamma^2 S a\beta^2 - 2S\beta\gamma S\gamma a S a\beta),$$

and accordingly, in terms of the sides of the triangle,

$$-Sa\beta\gamma = +(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)^{\frac{1}{2}}, \quad \text{(VII.)}$$

and thus the remaining fundamental relations are established.

Ex. 1. Prove that $$\alpha^{\frac{2A}{\pi}}\beta^{\frac{2B}{\pi}}\gamma^{\frac{2C}{\pi}} = -1,$$
rotation round α from β to γ being supposed positive.

[For the supplemental triangle $\frac{\beta'}{\alpha'} \cdot \frac{\alpha'}{\gamma'} \cdot \frac{\gamma'}{\beta'} = 1$, $\frac{\beta'}{\alpha'} = \gamma^{2-\frac{2C}{\pi}}$, etc. (compare Art. 29 (v.)).]

Ex. 2. Deduce the relations

$$\cos C + \cos A \cos B = \cos c \sin A \sin B,$$

$$\gamma \sin C = \alpha \sin A \cos B + \beta \sin B \cos A + V\alpha\beta \sin A \sin B.$$

Ex. 3. If P is any point on the surface of the sphere and Q the foot of the perpendicular let fall from this point on the side AB, prove that

$$\cos PC \sin C = \cos PA \sin A \cos B + \cos PB \sin B \cos A + \sin PQ \sin c \sin A \sin B.$$

Ex. 4. Taking P at the centre of the circumscribing small circle, prove that

$$2 \cot R \sin \tfrac{1}{2} \Sigma = \sin A \sin B \sin c,$$

where R is the radius of the small circle and where Σ is the spherical excess.

Ex. 5. Show how to represent versors and their products by versor angles analogous to the versor arcs of Art. 21, p. 16.

[By Ex. 1, $\gamma^{\frac{2}{\pi}(\pi - C)} = \alpha^{\frac{2A}{\pi}}\beta^{\frac{2B}{\pi}}$, so that if the versor $\alpha^{\frac{2A}{\pi}}$ is represented by a *directed angle* A at the extremity of the vector α, and if $\beta^{\frac{2B}{\pi}}$ is similarly represented by a directed angle B at the extremity of β; the product is represented by the *directed external angle* $\pi - \text{C}$ at the extremity of γ.

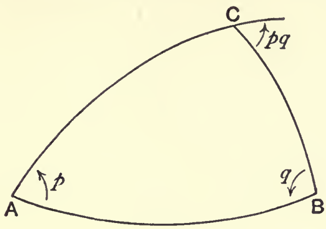

FIG. 21.

To construct the product of two versors p and q on this plan, let A be the extremity of UVp, and B of UVq. Draw the great circle AB, and the great circles AC and BC making the angles $\angle p$ and $\angle q$ with AB, and intersecting in the point C, round which rotation from A to B is positive. Then pq is represented by the external angle at C. To construct the product qp, a point $\text{C}_,$ must be similarly found below AB, so that rotation round it from B to A is positive. The method may be extended to spherical polygons (*Elements of Quaternions*, Art. 313).

ART. **31.** In his fifth and sixth lectures and in Art. 297 of the *Elements of Quaternions*, Hamilton has developed at considerable length a curious and interesting theory connected with the "fourth proportional" $\beta_,\alpha_,^{-1}\gamma_,$ to three given vectors and with the area of a spherical triangle ABC, whose sides are bisected in $\text{A}_,$, $\text{B}_,$ and $\text{C}_,$ by the extremities of these vectors.

The vectors α, β, and γ terminating at the vertices of ABC, and

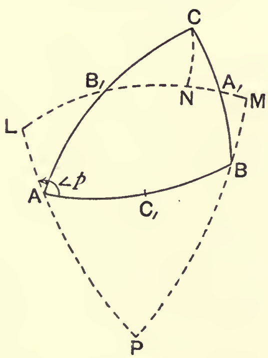

FIG. 22.

$\text{A}_,$, $\text{B}_,$, $\text{C}_,$ being the middle points of the sides of the triangle, we have the relations,

$$\frac{\alpha_,}{\beta}=\frac{\gamma}{\alpha_,}=\left(\frac{\gamma}{\beta}\right)^{\frac{1}{2}};\quad \frac{\beta_,}{\gamma}=\frac{\alpha}{\beta_,}=\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}};\quad \frac{\gamma_,}{\alpha}=\frac{\beta}{\gamma_,}=\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}};\ \ldots\ldots\ldots\ldots\text{(I.)}$$

and from these relations or directly, we find

$$\gamma = \alpha_{\prime}\beta\alpha_{\prime}^{-1}, \ \alpha = \beta_{\prime}\gamma\beta_{\prime}^{-1}, \ \beta = \gamma_{\prime}\alpha\gamma_{\prime}^{-1}. \ \ldots\ldots\ldots\ldots\ldots \text{(II.)}$$

Hence $\alpha = \beta_{\prime}\alpha_{\prime}\gamma_{\prime}\alpha\gamma_{\prime}^{-1}\alpha_{\prime}^{-1}\beta_{\prime}^{-1} = \beta_{\prime}\alpha_{\prime}^{-1}\gamma_{\prime}\alpha\gamma_{\prime}^{-1}\alpha_{\prime}\beta_{\prime}^{-1} = p\alpha p^{-1}$

if $\qquad p = \beta_{\prime}\alpha_{\prime}^{-1}\gamma_{\prime} \qquad$ (III.)

is the "fourth proportional" to $\beta_{\prime}$, $\alpha_{\prime}$ and $\gamma_{\prime}$, so that the conical rotation produced by $p(\ )p^{-1}$ leaves the vector α unchanged, and therefore $\pm\alpha$ is the axis of the quaternion p.

Again we have

$$p\gamma_{\prime}p^{-1} = \beta_{\prime}\alpha_{\prime}^{-1} \cdot \gamma_{\prime} \cdot \alpha_{\prime}\beta_{\prime}^{-1},$$

so that the conical rotation in question produces the same effect on the vector $\gamma_{\prime}$ as the conical rotation round P—the pole of the great circle A,B,—through twice the angle of $\beta_{\prime}\alpha_{\prime}^{-1}$. And because the point C, can be converted into the extremity of $p\gamma_{\prime}p^{-1}$ by a rotation round P or round A, this extremity must be the reflection of C, with respect to the great circle PA. Thus the angle of the quaternion p is C,AL if $+\alpha$ is its axis, while it is C,AP if $-\alpha$ is its axis, and we proceed to show that the former alternative is true.

The point P being the pole of A,B,, the angles L and M are right. Taking CN perpendicular to A,B, it follows that the triangle NCB, is equal to LAB, and that NCA, is equal to MBA,, for NCB, has the side B,C, the angle CB,N and the right angle CNB, equal respectively to the side AB,, the angle AB,L and the right angle ALB, of the triangle ALB,. Hence AL is equal to BM, both being equal to CN; the triangle APB is isosceles, its equal sides being complements of AL or BM; and the equal external angles C,AL or C,BM of this triangle are equal to $\frac{1}{2}(A+B+C)$, C,AL + C,BM being A + B + B,AL + A,BM = A + B + B,CN + A,CN. Moreover, if we join PC,, the angle PC,A will be right, C, being the middle of the base of the isosceles triangle APB; and the angle C,PA will be equal to $\angle\beta_{\prime}\alpha_{\prime}^{-1}$, for it is $\frac{1}{2}\angle$BPA or $\frac{1}{2}$ML or A,B,, since by the equality of the small triangles MA, = A,N and NB, = B,L. Hence by the construction of Ex. 5, Art. 30, the angle C,PA represents $\beta_{\prime}\alpha_{\prime}^{-1}$ and AC,P represents $\gamma_{\prime}$, so that C,AL represents p or $\beta_{\prime}\alpha_{\prime}^{-1}\gamma_{\prime}$, and therefore

$$\angle p = \angle\beta_{\prime}\alpha_{\prime}^{-1}\gamma_{\prime} = \tfrac{1}{2}(A+B+C), \ \mathrm{UV}p = \alpha. \ \ldots\ldots\ldots\ldots \text{(IV.)}$$

Again we have this remarkable transformation by (I.),

$$p\alpha^{-1} = \beta_{\prime}\alpha_{\prime}^{-1}\gamma_{\prime}\alpha^{-1} = \frac{\beta_{\prime}}{\gamma} \cdot \frac{\gamma}{\alpha_{\prime}} \cdot \frac{\gamma_{\prime}}{\alpha} = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}\left(\frac{\gamma}{\beta}\right)^{\frac{1}{2}}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}, \ \ldots\ldots\ldots \text{(V.)}$$

so that for the new quaternion,

$$p'=\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}\left(\frac{\gamma}{\beta}\right)^{\frac{1}{2}}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}, \quad\text{(VI.)}$$

$$\angle p'=\tfrac{1}{2}\Sigma=\tfrac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C}-\pi),\ \mathrm{UV}p'=\alpha, \quad\text{(VII.)}$$

if Σ is the *spherical excess* of the triangle ABC, because

$$\angle p'=\angle p\alpha^{-1}=\angle p-\frac{\pi}{2}.$$

EXAMPLES TO CHAPTER IV.

Ex. 1. If α is a unit vector at right angles to β, show that

$$\alpha^u\beta=\beta\alpha^{-u},$$

where u is a scalar.

Ex. 2. If α, β and γ are unit vectors, mutually at right angles,

$$\alpha^u\beta=\mathrm{V}\gamma^u+\mathrm{V}\beta^{u+1}.$$

Ex. 3. Given two sets α, β, γ and α', β', γ' of mutually rectangular unit vectors in the same order of rotation, so that $\alpha'=+\beta'\gamma'$ if $\alpha=+\beta\gamma$, show that we may connect the two sets by the series of relations

(1) $\gamma_1=\gamma, \quad \alpha_1=\alpha\cos\psi+\beta\sin\psi, \quad \beta_1=-\alpha\sin\psi+\beta\cos\psi$;

(2) $\beta_2=\beta_1, \quad \gamma_2=\gamma_1\cos\theta+\alpha_1\sin\theta, \quad \alpha_2=-\gamma_1\sin\theta+\alpha_1\cos\theta$;

(3) $\gamma'=\gamma_2, \quad \alpha'=\alpha_2\cos\phi+\beta_2\sin\phi, \quad \beta'=-\alpha_2\sin\phi+\beta_2\cos\phi$;

and draw a figure to exhibit the Eulerian angles ψ, θ and ϕ.

Ex. 4. The conical rotation $q(\ )q^{-1}$ which converts the first set of vectors of the last example into the second is determined by the versor

$$q=\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi+\psi)+\gamma\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi+\psi)$$
$$+\alpha\sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi-\psi)+\beta\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi-\psi)$$

(see Tait's *Quaternions*, Art. 373); while other expressions for the same versor are

$$q=\{(\gamma^{\frac{2\psi}{\pi}}\beta)^{\frac{2\theta}{\pi}}\gamma\}^{\frac{\phi}{\pi}}.(\gamma^{\frac{2\psi}{\pi}}\beta)^{\frac{\theta}{\pi}}.\gamma^{\frac{\psi}{\pi}}, \text{ and } q=\gamma^{\frac{\psi}{\pi}}\beta^{\frac{\theta}{\pi}}\gamma^{\frac{\phi}{\pi}}.$$

Ex. 5. Given in order n coinitial vectors $\alpha_1, \alpha_2, \ldots \alpha_n$, it is required to draw n planes, each through one of the vectors, so that the lines of intersection of each plane with the two adjacent may be equally inclined to the contained vector. Prove that the vector along the intersection of the planes through α_1 and α_n is parallel to $\mathrm{V}\alpha_n\alpha_{n-1}\ldots\alpha_1$.

Ex. 6. Show that

$$\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}\left(\frac{\gamma}{\beta}\right)^{\frac{1}{2}}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}=\frac{\mathrm{U}(\gamma+\alpha)}{\mathrm{U}(\beta+\gamma)}\cdot\frac{\mathrm{U}(\alpha+\beta)}{\alpha}=\frac{\alpha}{\mathrm{U}(\gamma+\alpha)}\cdot\frac{\mathrm{U}(\beta+\gamma)}{\mathrm{U}(\alpha+\beta)}$$
$$=\left(\frac{\gamma+\alpha}{\beta+\gamma}\cdot\frac{\alpha+\beta}{\gamma+\alpha}\cdot\frac{\beta+\gamma}{\alpha+\beta}\right)^{\frac{1}{2}},$$

where α, β and γ are any three unit vectors.

Ex. 7. If α, β, γ and δ are the vectors from the centre to four points A, B, C and D on a sphere of unit radius, show that

$$\alpha^{\frac{2A}{\pi}} \beta^{\frac{2B}{\pi}} \gamma^{\frac{2C}{\pi}} \delta^{\frac{2D}{\pi}} = 1,$$

when the quadrilateral is uncrossed, and when rotation round an internal point from A to B to C to D is positive.

(*a*) Hence

$$(\cos A + \alpha \sin A)(\cos B + \beta \sin B) = (\cos D - \delta \sin D)(\cos C - \gamma \sin C).$$

(*b*) Also

$$\cos A \cos B - \sin A \sin B \cos AB = \cos D \cos C - \sin D \sin C \cos CD\,;$$

and if P is any fifth point on the sphere from which perpendiculars PQ and PR are let fall on the arcs AB and CD,

$$\sin A \cos B \cos AP + \cos A \sin B \cos BP + \sin A \sin B \sin AB \sin PQ$$
$$+ \sin C \cos D \cos CP + \cos C \cos D \cos DP + \sin C \sin D \sin CD \sin PR = 0.$$

(*c*) Examine the cases in which P is taken to be the pole of a side or of a diagonal, or the point of intersection of AB and CD. (See *Elements of Quaternions*, Art. 313.)

Ex. 8. If $\alpha' = \mathrm{UV}\beta\gamma$, $\beta' = \mathrm{UV}\gamma\alpha$, $\gamma' = \mathrm{UV}\alpha\beta$, where $\mathrm{T}\alpha = \mathrm{T}\beta = \mathrm{T}\gamma = 1$, and $\mathrm{S}\alpha\beta\gamma < 0$, prove that $\alpha = \mathrm{UV}\beta'\gamma'$, $\beta = \mathrm{UV}\gamma'\alpha'$ and $\gamma = \mathrm{UV}\alpha'\beta'$.

(*a*) If A, B and C are the supplements of the angles between the pairs of vectors β', γ' ; γ', α' ; and α', β', deduce the relation

$$\alpha^{\frac{2A}{\pi}} \beta^{\frac{2B}{\pi}} \gamma^{\frac{2C}{\pi}} = -1.$$

(*b*) Show that this equation may be transformed into

$$e^{A\alpha} . e^{B\beta} . e^{C\gamma} = -1.$$

(*c*) Examine whether it may be further simplified to

$$e^{A\alpha + B\beta + C\gamma} = -1,$$

and carefully state your reason. (Bishop Law's Premium, 1898.)

CHAPTER V.

GEOMETRY OF THE STRAIGHT LINE AND PLANE.

ART. **32**. The vector $\rho = \text{OP}$ being drawn from a fixed origin and being regarded as variable, the equations

$$S\rho\alpha = 0, \text{ and } V\rho\beta = 0, \quad \text{(I.)}$$

represent respectively the plane through the origin perpendicular to α and the line through the origin parallel to β.

If $\gamma = \text{OC}$, $\delta = \text{OD}$, the equations of a plane through C and a line through D are respectively

$$S(\rho - \gamma)\alpha = 0, \text{ and } V(\rho - \delta)\beta = 0. \quad \text{(II.)}$$

These may be replaced by

$$\rho = \gamma + \alpha\tau, \text{ and } \rho = \delta + \beta t, \quad \text{(III.)}$$

where τ is an arbitrary vector subject to the single implied condition $S\alpha\tau = 0$, and where t is an arbitrary scalar.

The point E in which the line intersects the plane is the extremity of the vector,

$$\epsilon = \delta - \beta\frac{S(\delta - \gamma)\alpha}{S\beta\alpha} \text{ or } \epsilon = \gamma + \frac{V\alpha V(\gamma - \delta)\beta}{S\alpha\beta}. \quad \text{(IV.)}$$

The first of these expressions has been found by substituting $\delta + \beta t$ for ρ in the first equation (II.) of the plane. The second has been found by replacing ρ by $\gamma + \alpha\tau$ in the first equation of the line. Another expression for the vector to the same point of intersection is

$$\epsilon = \frac{\beta S\alpha\gamma + V\alpha V\beta\delta}{S\alpha\beta}. \quad \text{(V.)}$$

From (IV.) we have the intercept $\text{DE} = \epsilon - \delta$ on the line, and the interval $\text{CE} = \epsilon - \gamma$ in the plane between the fixed points and the point of intersection.

If we make in (IV.), $\beta=\alpha$, we find the foot of the perpendicular from the point D on the plane to be at the extremity of the vector

$$\text{OM}=\mu=\delta-\alpha^{-1}\text{S}(\delta-\gamma)\alpha \text{ or } \mu=\gamma+\alpha^{-1}\text{V}(\gamma-\delta)\alpha, \ldots\text{(VI.)}$$

since the line being now parallel to α is perpendicular to the plane.

The *vector perpendicular* from the point D on the plane is

$$\text{DM}=\mu-\delta=-\alpha^{-1}\text{S}(\delta-\gamma)\alpha=\alpha\text{S}\alpha^{-1}\text{DC}, \ldots\ldots\ldots\text{(VII.)}$$

and it will be noticed that we may directly obtain the vectors DM and CM by resolving the vector DC along and perpendicularly the vector α. (Art. 27.)

If in (IV.) we replace α by β, we find the foot of the perpendicular from the point C on the line to be the extremity of the vector

$$\text{ON}=\nu=\delta-\beta^{-1}\text{S}(\delta-\gamma)\beta \text{ or } \nu=\gamma+\beta^{-1}\text{V}(\gamma-\delta)\beta, \text{ (VIII.)}$$

because now the plane is perpendicular to the line. The vector perpendicular is

$$\text{CN}=\beta^{-1}\text{V}(\gamma-\delta)\beta=\beta^{-1}\text{V}\beta\text{CD}. \ldots\ldots\ldots\ldots\ldots\text{(IX.)}$$

In general the normal to the plane (II.) makes with the line an angle determined by

$$\cos\theta=\text{SU}\frac{\beta}{\alpha}, \text{ or } \sin\theta=\text{TVU}\frac{\beta}{\alpha}, \text{ or } \tan\theta=-\frac{\text{TV}\beta\alpha}{\text{SV}\beta\alpha}; \ldots\text{(X.)}$$

and if we are required to draw a plane through the point C making a given angle with the line, we have

$$\text{U}\beta=\cos\theta\text{U}\alpha+\sin\theta\text{U}\tau\alpha; \text{ while } \text{U}\alpha=\cos\theta\text{U}\beta+\sin\theta\text{U}\tau\beta, \ldots\text{(XI.)}$$

if the line is to be drawn inclined at a given angle to the plane. In these equations the vector τ is arbitrary, subject to the implied conditions, which are $\text{S}\tau\alpha=0$ and $\text{S}\tau\beta=0$ respectively.

Ex. 1. Two objects, B and C, are observed from the origin of the vector α to be in the directions $\text{U}\beta$ and $\text{U}\gamma$, and from the extremity of α to be in the directions $\text{U}\beta'$ and $\text{U}\gamma'$; prove that the vector BC is

$$\text{U}\gamma\cdot\frac{\text{V}\alpha\text{U}\gamma'}{\text{VU}\gamma\gamma'}-\text{U}\beta\cdot\frac{\text{V}\alpha\text{U}\beta'}{\text{VU}\beta\beta'},$$

and point out the conditions implied in this expression.

[For the point B we have $x\text{U}\beta=\alpha+y\text{U}\beta'$, and therefore

$$x\text{VU}\beta\beta'=\text{V}\alpha\text{U}\beta'.]$$

Ex. 2. Four points A, B, C, D are viewed from a fifth point P. Prove that they appear to form a parallelogram ABCD if

$$\text{U}(\text{UPA}+\text{UPC})=\text{U}(\text{UPB}+\text{UPD});$$

a rectangle if $\text{UPA}+\text{UPC}=\text{UPB}+\text{UPD}$;

and a square if in addition $\text{SUPA}\,.\,\text{PB}=\text{SUPB}\,.\,\text{PC}$.

[The first condition requires the diagonals AC and BD to appear to bisect one another. The second requires that they should also appear to be equal, and the third imposes the additional condition that adjacent sides should appear to be equal.]

Ex. 3. Find the equation of the locus of a point equidistant (1) from two fixed points, (2) from two fixed planes.

Ex. 4. The extremity of the vector ρ is projected from the extremity of the vector α into a point on the plane $S\lambda\rho+1=0$. Prove that this point lies at the extremity of the vector

$$\frac{V\lambda V\alpha\rho+(\rho-\alpha)}{S\lambda(\alpha-\rho)}.$$

ART. **33.** The equation of a plane through the points C, C′, and of a line through D, D′, are respectively,

$$S(\rho-\gamma)(\gamma'-\gamma)\alpha=0 \quad\text{and}\quad V(\rho-\delta)(\delta'-\delta)=0\,; \;\ldots\ldots\ldots\text{(I.)}$$

or $$S(\rho\gamma+\gamma\gamma'+\gamma'\rho)\alpha=0 \quad\text{and}\quad V(\rho\delta+\delta\delta'+\delta'\rho)=0\,; \;\ldots\ldots\text{(II.)}$$

or $$\rho=\frac{\gamma+s\gamma'}{1+s}+u\alpha \quad\text{and}\quad \rho=\frac{\delta+t\delta'}{1+t}\,; \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(III.)}$$

the plane being determined by the condition that the vectors CP and CC′ shall be coplanar with some fixed vector α, and the line requiring that DP shall be parallel to DD′.

The various expressions given in the last article may be modified to suit the present case by replacing α and β by $V(\gamma'-\gamma)\alpha$ and $\delta'-\delta$ respectively.

The plane through CC′ parallel to the line DD′ is

$$S(\rho-\gamma)(\gamma'-\gamma)(\delta'-\delta)=0, \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(IV.)}$$

because the normal to the plane must be perpendicular to the line, so that $SV(\gamma'-\gamma)\alpha\,.\,(\delta'-\delta)=0$, or $\alpha=x(\gamma'-\gamma)+y(\delta'-\delta)$, where x and y are certain scalars which disappear on substituting in (I.).

If a plane can be drawn through CC′ perpendicular to DD′, the equation $VV(\gamma'-\gamma)\alpha\,.\,(\delta'-\delta)=0$, requiring $S(\gamma'-\gamma)(\delta'-\delta)=0$, must be satisfied.

We may, without loss of generality, take α to be perpendicular to CC′, and as it easily appears that the plane for which in addition $S\alpha(\delta'-\delta)=0$ is most inclined to the given line, we can verify that the minimum value of

$$\tan\theta=-\frac{TV\,.\,V(\gamma'-\gamma)\alpha\,.\,(\delta'-\delta)}{S\,.\,V(\gamma'-\gamma)\alpha\,.\,(\delta'-\delta)} \text{ is } \tan\theta_0=-\frac{S(\gamma'-\gamma)(\delta'-\delta)}{TV(\gamma'-\gamma)(\delta'-\delta)}\text{(V.)}$$

where the vector α is regarded as variable, and that the plane

$$SV(\rho-\gamma)(\gamma'-\gamma)V(\gamma'-\gamma)(\delta'-\delta)=0 \;\ldots\ldots\ldots\ldots\text{(VI.)}$$

is most inclined to the given line.

ART. **34.** The equation of a plane through three given points, A, B, C, is

$$S\rho V(\beta\gamma+\gamma\alpha+\alpha\beta)=S\alpha\beta\gamma, \quad\text{(I.)}$$

for the condition that PA, PB and PC should be coplanar reduces to this expression; and in this equation $V(\beta\gamma+\gamma\alpha+\alpha\beta)$ represents double the vector area of the face ABC, while $-S\alpha\beta\gamma$ is the volume of the parallelepiped having three conterminous sides, OA, OB, OC (Art. 24). The equation may be taken as asserting that if through the boundary of a vector area determined by $V(\beta\gamma+\gamma\alpha+\alpha\beta)$ we draw vectors equal and parallel to OP (P being any point in the plane), the volume of the solid thus constructed is equal to that of the parallelepiped (Art. 23).

Writing for brevity, the equation of a plane in the form

$$S\lambda\rho=1, \quad\text{(II.)}$$

the vectors

$$\mu=\delta-\lambda^{-1}(S\lambda\delta-1)=\lambda^{-1}V\lambda\delta+\lambda^{-1}, \text{ and } DM=\lambda^{-1}-\lambda^{-1}S\lambda\delta \quad\text{(III.)}$$

are respectively the vector to the foot of the perpendicular from a point D on the plane, and the vector-perpendicular from the same point.

To find a plane equally inclined to three given lines OA, OB and OC, we have

$$\cos\theta\,.\,T\lambda=-S\lambda U\alpha=-S\lambda U\beta=-S\lambda U\gamma,$$

so that (Art. 26)

$$U\lambda\,.\,\sec\theta\,.\,SU\alpha\beta\gamma=-V(U\beta\gamma+U\gamma\alpha+U\alpha\beta),$$
$$\sec\theta=-TV(U\beta\gamma+U\gamma\alpha+U\alpha\beta)(SU\alpha\beta\gamma)^{-1},$$

and the equation of the plane is

$$S\rho V(U\beta\gamma+U\gamma\alpha+U\alpha\beta)=\textit{const.},$$

or

$$S\rho V(\beta\gamma T\alpha+\gamma\alpha T\beta+\alpha\beta T\gamma)=\textit{const.}$$

A plane equally inclined to the faces of the pyramid OABC is represented by

$$S\rho(\alpha TV\beta\gamma+\beta TV\gamma\alpha+\gamma TV\alpha\beta)=\textit{const.};$$

a plane cutting off equal areas on its faces is

$$S\rho(UV\beta\gamma+UV\gamma\alpha+UV\alpha\beta)=\textit{const.},$$

or

$$S\rho V\left(\frac{\beta\gamma}{TV\beta\gamma}+\frac{\gamma\alpha}{TV\gamma\alpha}+\frac{\alpha\beta}{TV\alpha\beta}\right)=\textit{const.};$$

while the equations of the planes cutting off equal intercepts from the edges and from the normals to the faces have been already found.

Ex. 1. Find a plane equally inclined to the bisectors of the angles of the faces of the pyramid OABC.

Ex. 2. The planes through an edge and through the bisector of the angle of the opposite face intersect in a line.

Ex. 3. Find the equation of the plane bisecting the angle between a pair of faces.

Ex. 4. Find the equation of a plane through an edge and normal to the opposite face, and prove that three such planes intersect in a line.

ART. **35.** The line of intersection of the planes

$$S\lambda\rho=1,\ \ S\mu\rho=1 \ \text{ is } \ V\rho V\lambda\mu=\mu-\lambda, \ \text{ or } \ \rho=\frac{\mu-\lambda+t}{V\lambda\mu}; \ldots\text{(I.)}$$

and that of the planes

$$S\lambda\rho=1,\ \ S\mu\rho=0 \ \text{ is } \ V\rho V\lambda\mu=\mu, \ \text{ or } \ \rho=\frac{\mu+t}{V\lambda\mu}.$$

Three planes $S\lambda\rho=l$, $S\mu\rho=m$, $S\nu\rho=n$ intersect in the point

$$\rho S\lambda\mu\nu=V(l\mu\nu+m\nu\lambda+n\lambda\mu); \ldots\ldots\ldots\ldots\ldots\text{(II.)}$$

and the condition that the planes should intersect in a line is

$$V(l\mu\nu+m\nu\lambda+n\lambda\mu)=0, \ldots\ldots\ldots\ldots\ldots\text{(III.)}$$

if l, m and n are not all zero. If they are all zero, the condition is

$$S\lambda\mu\nu=0. \ldots\ldots\ldots\ldots\ldots\text{(IV.)}$$

Four planes intersect in a point if the condition

$$S(l\mu\nu\varpi-m\lambda\nu\varpi+n\lambda\mu\varpi-p\lambda\mu\nu)=0 \ldots\ldots\ldots\ldots\text{(V.)}$$

is satisfied, the equation of the fourth plane being $S\rho\varpi=p$.

The conditions of intersection (III.) and (V.) may be replaced by the pairs of simultaneous equations

$$x\lambda+y\mu+z\nu=0, \ \ xl+ym+zn=0; \ldots\ldots\ldots\ldots\text{(VI.)}$$

and

$$x\lambda+y\mu+z\nu+w\varpi=0, \ \ xl+ym+zn+wp=0 \ldots\ldots\text{(VII.)}$$

respectively, the compatibility of the equations (VI.) or (VII.) being equivalent to (III.) or (V.).

ART. **36.** Given a pair of lines

$$V(\rho-\gamma)\alpha=0, \text{ or } \rho=\gamma+t\alpha; \text{ and } V(\rho-\gamma')\alpha'=0, \text{ or } \rho=\gamma'+t'\alpha', \text{(I.)}$$

the vector from a point P on the first to a point P′ on the second is

$$PP'=\gamma'-\gamma+t'\alpha'-t\alpha. \ldots\ldots\ldots\ldots\ldots\text{(II.)}$$

If it is possible to select the scalars t and t' so that this vector may vanish, the lines intersect and the condition of their intersection is

$$S.PP'V\alpha'\alpha=0, \ \text{ or } \ S(\gamma'-\gamma)\alpha'\alpha=0, \ldots\ldots\ldots\ldots\text{(III.)}$$

P and P′ being arbitrary points on the lines.

Resolving the vector PP′ into two components, parallel and perpendicular to the vector $V\alpha'\alpha$, which is at right angles to the directions of the two lines,

$$\begin{aligned}\mathrm{PP}' &= V\alpha\alpha' S(V\alpha\alpha')^{-1}\mathrm{PP}' + V\alpha\alpha' V\,.\,(V\alpha\alpha')^{-1}\mathrm{PP}' \\ &= V\alpha\alpha' S\frac{\mathrm{PP}'}{V\alpha\alpha'} + \alpha S\frac{\alpha'}{V\alpha\alpha'}\mathrm{PP}' - \alpha' S\frac{\alpha}{V\alpha\alpha'}\mathrm{PP}',\end{aligned}$$

and substituting from (II.) on the right,

$$\left.\begin{aligned}\mathrm{PP}' = V\alpha\alpha' S\frac{\gamma'-\gamma}{V\alpha\alpha'} + \alpha\left(S\frac{\alpha'}{V\alpha\alpha'}(\gamma'-\gamma) - t\right) \\ - \alpha'\left(S\frac{\alpha}{V\alpha\alpha'}(\gamma'-\gamma) - t'\right)\end{aligned}\right\} \quad \ldots\ldots\ldots(\mathrm{IV.})$$

Thus the line joining the arbitrary points has a fixed component perpendicular to the directions of the two lines, and suitably selecting the scalars t and t' in (IV.) we see that

$$\left.\begin{aligned}\mathrm{P_0P_0}' = V\alpha\alpha' S\frac{\gamma'-\gamma}{V\alpha\alpha'}, \quad \mathrm{OP_0} = \gamma + \alpha S\frac{\alpha'}{V\alpha\alpha'}(\gamma'-\gamma), \\ \mathrm{OP_0}' = \gamma' + \alpha' S\frac{\alpha}{V\alpha\alpha'}(\gamma'-\gamma)\end{aligned}\right\} \ldots\ldots(\mathrm{V.})$$

are respectively, the vector-perpendicular to the two lines, or the vector shortest distance from the first line to the second, and the vectors from the origin to the feet of this shortest vector—the points $\mathrm{P_0}$ and $\mathrm{P_0}'$.

Ex. 1. Verify that $\mathrm{P_0P_0}' = \mathrm{OP_0}' - \mathrm{OP_0}$ in equation (V.).

Ex. 2. Draw a line through a point (E) to intersect two given lines $V(\rho-\gamma)\alpha = 0$, $V(\rho-\gamma')\alpha' = 0$.

[The line is parallel to $V\,.\,V(\epsilon-\gamma)\alpha V(\epsilon-\gamma')\alpha'$. See (III.).]

Ex. 3. The locus of a line which intersects three given lines is represented by

$$S\,.\,V(\rho-\gamma)\alpha V(\rho-\gamma')\alpha' V(\rho-\gamma'')\alpha'' = 0.$$

(a) Reduce this equation to the form $XY = ZW$, where X, Y, Z and W are planes.

Ex. 4. Writing $\qquad \sigma = V\rho_1\rho_2, \quad \tau = \rho_2 - \rho_1,$

prove that σ and τ are merely multiplied by a scalar, if for ρ_1 and ρ_2 are substituted the vectors to any two points on the line of their extremities.

(a) Conversely, given any two vectors, σ and τ, satisfying the relation $S\sigma\tau = 0$, show how they determine a line parallel to τ.

(b) In this notation any two lines may be denoted by the symbols (σ, τ) and (σ', τ'). Prove that the lines intersect if

$$S\sigma\tau' + S\sigma'\tau = 0.$$

(c) Any scalar relation homogeneous in the pair of vectors σ and τ imposes a single condition on a line.

(d) If the planes $S\lambda_1\rho+1=0$, $S\lambda_2\rho+1=0$ contain the extremities of the vectors ρ_1 and ρ_2, show that

$$\sigma=V\rho_1\rho_2=u(\lambda_2-\lambda_1),\quad \tau=\rho_2-\rho_1=-uV\lambda_1\lambda_2,$$

where u is some scalar.

(e) Hence any relation homogeneous in the pair of vectors σ and τ when equated to zero may be expressed in the forms

$$f(\sigma,\tau)=0,\quad f(V\rho_1\rho_2,\rho_2-\rho_1)=0,\quad f(\lambda_2-\lambda_1,-V\lambda_1\lambda_2)=0.$$

(f) According as the equation $f(\sigma,\tau)=0$

is equivalent to one, two or three scalar equations, it represents a complex, a congruence or a regulus of right lines, and the constituents of the vectors σ and τ, when resolved along three mutually rectangular directions, are Plücker's coordinates of a line. (See Salmon, *Geometry of Three Dimensions*, Chap. XIII., Section II.)

(g) The lines of a complex $f(\sigma,\tau)=0$ (f being now a scalar function), which pass through a point, the extremity of the fixed vector ρ_1, generate a cone

$$f(V\rho_1\tau,\tau)=0;$$

and the lines which lie in a fixed plane, $S\lambda_1\rho+1=0$, envelope the cone whose vertex is the origin and which is the reciprocal of the cone

$$f(\sigma,-V\lambda_1\sigma)=0.$$

ART. **37.** The vector to any point on the line joining two given points A and B is

$$OP=\rho=\frac{\alpha+t\beta}{1+t}, \qquad\text{(I.)}$$

t being a variable scalar. If P_1 and P_2 are any two points on the line, their vector distance is

$$P_1P_2=\frac{\alpha+t_2\beta}{1+t_2}-\frac{\alpha+t_1\beta}{1+t_1}=\frac{(t_2-t_1)(\beta-\alpha)}{(1+t_1)(1+t_2)}=\frac{(t_2-t_1)AB}{(1+t_1)(1+t_2)}; \qquad\text{(II.)}$$

and the anharmonic ratio of any four collinear points is

$$(P_1P_2P_3P_4)=\frac{P_1P_2\,.\,P_3P_4}{P_2P_3\,.\,P_4P_1}=\frac{(t_2-t_1)(t_4-t_3)}{(t_3-t_2)(t_1-t_4)}. \qquad\text{(III.)}$$

In particular

$$(APBP')=\frac{(t-0)(t'-\infty)}{(\infty-t)(0-t')}=\frac{t}{t'}. \qquad\text{(IV.)}$$

More generally, the anharmonic ratio of any four points Q_1, Q_2, Q_3 and Q_4 collinear with any two points P′, P″, of the range,

$$OQ=\frac{aOP'+tbOP''}{a+tb},\ \text{is}\ (Q_1Q_2Q_3Q_4)=\frac{(t_2-t_1)(t_4-t_3)}{(t_3-t_2)(t_1-t_4)}. \qquad\text{(V.)}$$

The two ranges (I.) and (V.) are homographic.

Ex. 1. If the range APBP′ is harmonic, prove that

$$\frac{1}{AP}+\frac{1}{AP'}=\frac{2}{AB},\quad\text{or}\quad \frac{1}{\rho-\alpha}+\frac{1}{\rho'-\alpha}=\frac{2}{\beta-\alpha}.$$

Ex. 2. Any two homographic ranges situated on a common line,

$$\rho=\frac{a\alpha+tb\beta}{a+tb},\quad \rho'=\frac{c\gamma+td\delta}{c+td},$$

may be simultaneously reduced to the forms,

$$\rho=\frac{\epsilon+s\eta}{1+s},\quad \rho'=\frac{e\epsilon+s\eta}{e+s},$$

Ex. 3. Show that the vectors ϵ and η satisfy the equation,

$$ad(\alpha-\epsilon)(\delta-\epsilon)-bc(\beta-\epsilon)(\gamma-\epsilon)=0.$$

ART. **38.** In many problems relating to a tetrahedron, it is convenient to have the equations expressed in a symmetrical manner, and some of the following relations will be found occasionally useful.

If the vectors λ, μ, ν and ϖ are the vector areas of the faces of a tetrahedron ABCD we may write

$$\left.\begin{array}{ll}\lambda=\mathrm{V}(\beta\gamma+\gamma\delta+\delta\beta), & \mu=-\mathrm{V}(\alpha\gamma+\gamma\delta+\delta\alpha),\\ \nu=\mathrm{V}(\alpha\beta+\beta\delta+\delta\alpha), & \varpi=-\mathrm{V}(\alpha\beta+\beta\gamma+\gamma\alpha).\end{array}\right\}\text{..............(I.)}$$

These vectors are independent of the origin, and their sum is zero, or

$$\Sigma\lambda=\lambda+\mu+\nu+\varpi=0. \text{..............................(II.)}$$

Again, if l, m, n, p are the sextupled volumes of the pyramids subtended at the origin by the four faces,

$$l=\mathrm{S}\beta\gamma\delta,\quad m=-\mathrm{S}\alpha\gamma\delta,\quad n=\mathrm{S}\alpha\beta\delta,\quad p=-\mathrm{S}\alpha\beta\gamma\,; \text{............(III.)}$$

and their sum is the sextupled volume of the tetrahedron, or

$$\Sigma l=l+m+n+p=v, \text{..............................(IV.)}$$

and is independent of the origin. Also,

$$\Sigma l\alpha=l\alpha+m\beta+n\gamma+p\delta=0. \text{............................(V.)}$$

Changing the origin to the extremity of the vector ω, and putting $\alpha'=\alpha-\omega$, etc., the volumes subtended by the faces at the new origin are

$$l'=\mathrm{S}\beta'\gamma'\delta'=\mathrm{S}(\beta-\omega)(\gamma-\omega)(\delta-\omega),\ \text{etc.},$$

or $$l'=l-\mathrm{S}\omega\lambda,\quad m'=m-\mathrm{S}\omega\mu,\quad n'=n-\mathrm{S}\omega\nu,\quad p'=p-\mathrm{S}\omega\varpi. \text{.........(VI.)}$$

But still (by v.),

$$\Sigma l'\alpha'=0=\Sigma(l-\mathrm{S}\omega\lambda)(\alpha-\omega)=\Sigma l\alpha+\omega\Sigma l-\Sigma\alpha\mathrm{S}\omega\lambda+\mathrm{S}\omega\Sigma\lambda,$$

and this reduces by former results to the new relation,

$$\omega\Sigma l+\Sigma\alpha\mathrm{S}\omega\lambda=0, \text{..............................(VII.)}$$

which holds for all vectors ω. Operating on this by $\mathrm{S}\omega'$, we may write the result in the form, $\mathrm{S}\omega(\omega'\Sigma l+\Sigma\lambda\mathrm{S}\alpha\omega')=0$; and, because ω is arbitrary, the part within brackets must vanish. But ω' is also arbitrary, and accordingly, for all vectors ω, we have

$$\omega\Sigma l+\Sigma\lambda\mathrm{S}\omega\alpha=0. \text{..............................(VIII.)}$$

Again, it is easy to see that

$$\Sigma\alpha\lambda=\alpha\lambda+\beta\mu+\gamma\nu+\delta\varpi=-3v=\Sigma\lambda\alpha\,; \text{..................(IX.)}$$

and, for verification, it is sufficient to take the terms in $\alpha\beta\gamma$, which are

$$\alpha\mathrm{V}\beta\gamma-\beta\mathrm{V}\alpha\gamma+\gamma\mathrm{V}\alpha\beta=-3p.$$

The sum $\Sigma\alpha\lambda$ is independent of the origin.

On the whole, we have

$$\Sigma\lambda=0\,;\ \Sigma l=v\,;\ \Sigma l\alpha=0\,;\ -\omega v=\Sigma\lambda\mathrm{S}\omega\alpha=\Sigma\alpha\mathrm{S}\omega\lambda\,;\ -3v=\Sigma\alpha\lambda=\Sigma\lambda\alpha. \text{...(X.)}$$

It is sometimes convenient to employ the vector perpendiculars from the vertices on the opposite faces instead of the vector areas. If $\alpha_{\prime}$, $\beta_{\prime}$, $\gamma_{\prime}$ and $\delta_{\prime}$ are these vectors, it is easily seen that

$$v=\alpha_{\prime}\lambda=\beta_{\prime}\mu=\gamma_{\prime}\nu=\delta_{\prime}\varpi, \qquad \text{(XI.)}$$

because, in fact, the equation of the face BCD may be written

$$S\rho\lambda=l, \text{ or } S(\rho-\alpha)\lambda=v, \text{ or } S(\rho-\alpha)\alpha_{\prime}^{-1}=1.$$

Thus (x.) gives

$$\Sigma\frac{1}{\alpha_{\prime}}=0,\ \Sigma l=v\ ;\ \Sigma l\alpha=0\ ;\ -\omega v=\Sigma\frac{1}{\alpha_{\prime}}S\omega\alpha=\Sigma\alpha S\frac{\omega}{\alpha_{\prime}}\ ;\ -3=\Sigma\frac{\alpha}{\alpha_{\prime}}=\Sigma\frac{1}{\alpha_{\prime}}\alpha. \qquad \text{(XII.)}$$

Ex. 1. Prove that the vector sides of the tetrahedron are given in terms of the vector areas of the faces by the relations

$$V\lambda\mu=(\gamma-\delta)v\ ;\ V\lambda\nu=-(\beta-\delta)v\ ;\ V\lambda\varpi=(\beta-\gamma)v\ ;$$

$$V\mu\nu=(\alpha-\delta)v\ ;\ V\mu\varpi=-(\alpha-\gamma)v\ ;\ V\nu\varpi=(\alpha-\beta)v\ ;$$

and show how to connect the rule of signs with that for the expansion of a determinant of the fourth order.

Ex. 2. Show that

$$S\mu\nu\varpi=S\lambda\nu\varpi=S\lambda\mu\varpi=-S\lambda\mu\nu=v^2.$$

Ex. 3. Given the magnitudes of the areas of the faces of a tetrahedron, show that the directions of the normals $U\lambda$, $U\mu$, and $U\nu$ to three of the faces must satisfy the relation

$$T\varpi^2=T\lambda^2+T\mu^2+T\nu^2-2T\mu\nu SU\mu\nu-2T\nu\lambda SU\nu\lambda-2T\lambda\mu SU\lambda\mu.$$

ART. **39.** Any five vectors are connected by relations of the form

$$a\alpha+b\beta+c\gamma+d\delta+e\epsilon=0, \text{ where } a+b+c+d+e=0\ ; \qquad \text{(I.)}$$

and if the vectors are drawn from a common origin O, and terminate at the five points A, B, C, D, E,

$$a:b:c:d:e=(\text{BCDE}):-(\text{ACDE}):(\text{ABDE}):-(\text{ABCE}):(\text{ABCD}), \qquad \text{(II.)}$$

where (ABCD) is the volume of the tetrahedron determined by the four points A, B, C, D.

To prove this, remark that if

$$a(\alpha-\epsilon)+b(\beta-\epsilon)+c(\gamma-\epsilon)+d(\delta-\epsilon)=0,$$

the ratios of the four scalars a, b, c and d have the values defined by equation (II.). (Compare Art. 24, Ex. 5.) The fifth scalar e is $-(a+b+c+d)$.

It should be noticed that the five scalars are absolutely independent of the origin of vectors.

Ex. Any five quaternions are connected by a relation of the form

$$xp+yq+zr+ws+vt=0$$

where x, y, z, w and v are scalars.

ART. **40.** Hamilton has elaborated a remarkable system of coordinates which he terms "Anharmonic Coordinates," the nature of which we proceed to explain.

In accordance with the last Article we may write any vector OP in terms of the vectors to four points A, B, C, D in the form

$$\text{OP}=\rho=\frac{xa\alpha+yb\beta+zc\gamma+wd\delta}{xa+yb+zc+wd} \qquad \text{(I.)}$$

where a, b, c and d are arbitrarily assumed constants and where x, y, z and w are the anharmonic coordinates in question.

The point U at the extremity of the vector

$$\text{OU}=\upsilon=\frac{a\alpha+b\beta+c\gamma+d\delta}{a+b+c+d}, \quad\text{(II.)}$$

is called the unit point, its anharmonic coordinates being equal to unity.

The point $\text{P}_{\prime\prime}$ whose coordinates are $x+tx'$, $y+ty'$, $z+tz'$, $w+tw'$, is collinear with the points P and P$'$, for

$$\text{OP}_{\prime}=\frac{\text{OP}\Sigma xa+\text{OP}'t\Sigma x'a}{\Sigma xa+t\Sigma xa}. \quad\text{(III.)}$$

And, in particular, the planes CDP and CDU cut the edge AB in the points determined by

$$\text{OP}_{12}=\frac{xa\alpha+yb\beta}{xa+yb}, \quad \text{OU}_{12}=\frac{a\alpha+b\beta}{a+b}, \quad\text{(IV.)}$$

for P_{12}, P and P_{34} are collinear, and also U_{12}, U and U_{34}, where

$$\text{OP}_{34}=\frac{zc\gamma+wd\delta}{zc+wd}, \quad \text{OU}_{34}=\frac{c\gamma+d\delta}{c+d}.$$

Denoting by (CD . APBU) the anharmonic ratio of the pencil of planes through the edge CD and the points A, P, B and U, we have

$$(\text{CD . APBD})=(\text{AP}_{12}\text{BU}_{12})=\frac{y}{x}; \quad\text{(V.)}$$

and similarly,
$$(\text{AC . BPDU})=\frac{w}{y}, \text{ etc.}$$

The ratios consequently of pairs of the coordinates, x, y, z, w of a point P are expressible as anharmonic ratios; and *the coordinates are unchanged by any linear transformation*, it being understood that the unit point undergoes the same transformation as the vertices of the tetrahedron.

To suit special circumstances, the unit point may be specially selected. It may, for example, be taken at the mean point of the tetrahedron, and then $a=b=c=d=1$.

Ex. 1. The vector ρ of any point P of space may, in indefinitely many ways, be expressed under the form

$$\text{OP}=\rho=\frac{xa\alpha+yb\beta+zc\gamma+wd\delta+ve\epsilon}{xa+yb+zc+wd+ve},$$

where
$$a\alpha+b\beta+c\gamma+d\delta+e\epsilon=0, \quad a+b+c+d+e=0.$$

[In terms of the four vectors α, β, γ, δ, the anharmonic coordinates of the point are $x-v$, $y-v$, $z-v$ and $w-v$. See also Art. 39.]

Ex. 2. The equation of a plane in anharmonic coordinates being

$$lx+my+nz+pw=0,$$

prove that the ratios of the coordinates of the plane l, m, n, p are expressible as anharmonic ratios.

[The line AB cuts the plane in the point $\text{OL}_{12}=\dfrac{am\alpha-bl\beta}{am-bl}$, and the anharmonic ratio $(\text{AU}_{12}\text{BL}_{12})=-\dfrac{m}{l}$.]

Ex. 3. Find the condition that the planes l, m, n, p and l', m', n', p should be parallel.

[The plane at infinity is $ax+by+cz+dw=0$.]

EXAMPLES TO CHAPTER V.

Ex. 1. The equation of the plane through the origin perpendicular to the vector α may be written in any one of the seven forms,

$$S\frac{\rho}{\alpha}=0\,;\quad T\frac{\rho+\alpha}{\rho-\alpha}=1\,;\quad \angle\frac{\rho}{\alpha}=\frac{\pi}{2}\,;\quad \frac{\rho}{\alpha}+K\frac{\rho}{\alpha}=0\,;\quad U\left(\frac{\rho}{\alpha}\right)^2=-1\,;$$
$$T(\rho+\alpha)=T(\rho-\alpha)\ ;\quad S\rho\alpha=0.$$

Ex. 2. The equation
$$T(\rho-\alpha)=T(\rho-\beta)$$
represents the plane bisecting at right angles the line AB.

Ex. 3. The equations
$$U\frac{\rho}{\alpha}=1,\quad U\frac{\rho}{\alpha}=-1,\quad \left(U\frac{\rho}{\alpha}\right)^2=1$$
represent respectively the half-line through the origin, having the direction of the vector α, the half-line having the direction of $-\alpha$, and the whole line parallel to α.

Ex. 4. The equations
$$SU\frac{\rho}{\alpha}=SU\frac{\beta}{\alpha},\quad SU\frac{\rho}{\alpha}=-SU\frac{\beta}{\alpha}$$
represent the two sheets of the cone of revolution, with O for vertex, OA for axis, and passing through the point B (*Elements*, Art. 196 (4)).

Ex. 5. The equation
$$TV\frac{\rho}{\alpha}=TV\frac{\beta}{\alpha}$$
represents the right circular cylinder, of which OA is the axis and B a point.

Ex. 6. If A, B, C and D are the vertices of a regular tetrahedron having its centre at the origin,
$$\alpha+\beta+\gamma+\delta=0\,;$$
$$\alpha^2=\beta^2=\text{etc.}=-3S\alpha\beta=-3S\beta\gamma=\text{etc.}\,;$$
$$T\text{AB}=2\sqrt{\tfrac{2}{3}}\,T\text{OA}.$$

Ex. 7. Find the area of a face of the regular tetrahedron and the volume in terms of the vector from the centre to a vertex.

Ex. 8. The six vectors $\pm\alpha$, $\pm\beta$, $\pm\gamma$ terminate at the vertices of a regular octahedron. Find the conditions the vectors must satisfy, and determine the volume, area of face, length of side.

Ex. 9. If A, B, C, D are any four points in a plane, the vectors α, β, γ, δ, drawn from an arbitrary origin to terminate at these points, are connected by a relation of the form,
$$a\alpha+b\beta+c\gamma+d\delta=0,\quad \text{where}\quad a+b+c+d=0.$$

(*a*) The vector
$$\text{OC}'=\gamma'=\frac{a\alpha+b\beta}{a+b}=\frac{c\gamma+d\delta}{c+d}$$
terminates at the point of intersection of AB and CD.

(*b*) If A′ and B′ are points similarly constructed on the remaining sides BC and CA of the triangle ABC,
$$\frac{\text{AC}'}{\text{C}'\text{B}}=\frac{a}{b}\,;\quad \frac{\text{BA}'}{\text{A}'\text{C}}=\frac{b}{c}\,;\quad \frac{\text{CB}'}{\text{B}'\text{A}}=\frac{c}{a}.$$

(*c*) Hence deduce the equation of six segments,

$$\frac{\text{AC}'}{\text{C}'\text{B}}\cdot\frac{\text{BA}'}{\text{A}'\text{C}}\cdot\frac{\text{CB}'}{\text{B}'\text{A}}=1.$$

(*d*) The right line B′C′ meets BC in the point A″, where

$$\text{OA}''=\alpha''=\frac{b\beta-c\gamma}{b-c}=\frac{(a+b)\gamma'-(c+a)\beta'}{b-c}.$$

(*e*) Hence A′ and A″ are harmonic conjugates to B and C.

(*f*) The equation of the six segments made by the transversal C′B′A″ is

$$\frac{\text{AC}'}{\text{C}'\text{B}}\cdot\frac{\text{CB}'}{\text{B}'\text{A}}\cdot\frac{\text{BA}''}{\text{A}''\text{C}}=-1.$$

(*g*) The points A″, B″, C″ are collinear, and the vectors α'', β'' and γ'' are connected by a relation,

$$l\alpha''+m\beta''+n\gamma'',\quad\text{where}\quad l+m+n=0.$$

(*h*) The line AD meets B′C″ in the point A‴, where

$$\text{OA}'''=\alpha'''=\frac{a\alpha-d\delta}{a-d}=\frac{(a+b)\gamma'+(c+a)\beta'}{2a+b+c}.$$

(*i*) The points B‴, C‴, A″ lie on the polar line of the point A with respect to the triangle BCD.

Ex. 10. Let ABCD be any tetrahedron, and E any arbitrary point, the vectors from an arbitrary origin to the five points A, B, C, D, E are connected by the relation,

$$a\alpha+b\beta+c\gamma+d\delta+e\epsilon=0,\quad a+b+c+d+e=0.$$

(*a*) The line AE meets the opposite face in A′, where

$$\text{OA}'=\alpha'=\frac{a\alpha+e\epsilon}{a+e}=\frac{b\beta+c\gamma+d\delta}{b+c+d}.$$

(*b*) The line A′B′ intersects the line AB in the point,

$$\frac{a\alpha-b\beta}{a-b}.$$

(*c*) The six points formed in this way form a complete quadrilateral.

(*d*) The vector to any point in the plane of this quadrilateral is of the form,

$$\rho=\frac{x(a\alpha-b\beta)+y(a\alpha-c\gamma)+z(a\alpha-d\delta)+w(a\alpha+b\beta+c\gamma+d\delta+e\epsilon)}{x(a-b)+y(a-c)+z(a-d)+w(a+b+c+d+e)}.$$

(*e*) The line AE meets this plane in the point $\text{A}_{,}$, where

$$\text{OA}_{,}=\frac{4a\alpha+e\epsilon}{4a+e}.$$

Ex. 11. The tetrahedra whose vertices are at the extremities of the vectors α, β, γ, δ and $a\alpha$, $b\beta$, $c\gamma$, $d\delta$ respectively are in perspective.

(*a*) Corresponding edges intersect in points at the extremities of vectors of the type,

$$\frac{\alpha a(1-b)-\beta b(1-a)}{a-b}.$$

(*b*) The six points thus determined form a complete quadrilateral.

(*c*) Prove that the equation of the plane of perspective may be written in the form,

$$\Sigma\pm ab(c-d)\text{S}\rho\alpha\beta+\Sigma\pm(1-a)bcd\text{S}\beta\gamma\delta=0,$$

the determinant law of signs being obeyed.

Ex. 12. Determine a parallelepiped, having its vertices on the four lines joining the origin to the points A, B, C and D, and having its centre at the origin.

(*a*) If
$$\text{OP}=\rho=\frac{x\alpha+y\beta+z\gamma+w\delta}{x+y+z+w},$$
a parallelepiped having its centre at P and its vertices on the lines PA, PB, PC, PD, has its vertices at the extremities of the vectors,
$$\rho\pm x(\alpha-\rho),\ \ \rho\pm y(\beta-\rho),\ \ \rho\pm z(\gamma-\rho),\ \ \rho\pm w(\delta-\rho).$$

(*b*) If a pair of edges are at right angles, the condition may be written in either of the forms, $\quad S\beta'\gamma'=S\alpha'\delta'$ or $\beta'^2+\gamma'^2=\alpha'^2+\delta'^2$,

where, for brevity, $\alpha'=x(\alpha-\rho)$, etc.

(*c*) The locus of a point P satisfying this condition is a quartic surface.

(*d*) If two pairs of edges are at right angles, the conditions may be written as
$$\alpha'^2=\beta'^2,\ \ \gamma'^2=\delta'^2.$$

(*e*) If the parallepiped is rectangular, the conditions are
$$\alpha'^2=\beta'^2=\gamma'^2=\delta'^2.$$

(*f*) The point, or points, satisfying these conditions are also given by
$$U(\alpha-\rho)\pm U(\beta-\rho)\pm U(\gamma-\rho)\pm U(\delta-\rho)=0,$$
and it may be shown that this is the condition that
$$T(\alpha-\rho)\pm T(\beta-\rho)\pm T(\gamma-\rho)\pm T(\delta-\rho)$$
should be a minimum.

(*g*) Another form of this condition is
$$\begin{aligned}SU\,.(\rho-\beta)(\rho-\gamma)(\rho-\delta)&=\pm SU\,.(\rho-\alpha)(\rho-\gamma)(\rho-\delta)\\&=\pm SU\,.(\rho-\alpha)(\rho-\beta)(\rho-\delta)\\&=\pm SU\,.(\rho-\alpha)(\rho-\beta)(\rho-\gamma).\end{aligned}$$

Ex. 13. Find the vector to a point P at which the faces of a tetrahedron subtend volumes whose ratios are given.

Ex. 14. Find a vector equation for determining a point P at which the faces of a tetrahedron subtend solid angles whose sines are in a given ratio.

Ex. 15. What is the condition in terms of the lengths of the sides of a tetrahedron that two opposite edges should be at right angles to one another?

(*a*) If two pairs of opposite edges are at right angles, the third pair is also at right angles.

Ex. 16. The vectors α, β and γ are coinitial. It is required to draw through the extremity of α a plane which shall cut the vectors in points forming a triangle of given species. Show that the problem may be reduced to finding scalars y and z, so that
$$lT(y\beta-z\gamma)=mT(z\gamma-\alpha)=nT(\alpha-y\beta),$$
where l, m and n are given scalars; and eliminate either y or z, so as to obtain an equation in the uneliminated scalar.

Ex. 17. If the perpendiculars from the vertices of the tetrahedron ABCD intersect, and if the origin is at the points of intersection, show that
$$S\alpha\beta=S\alpha\gamma=S\alpha\delta=S\beta\gamma=S\beta\delta=S\gamma\delta.$$

Ex. 18. Given three points A, B, C, show that the three equations

$$S(\rho-\alpha)(\beta-\gamma)=0,\quad S(\rho-\beta)(\gamma-\alpha)=0,\quad S(\rho-\gamma)(\alpha-\beta)=0$$

represent a line which is the locus of the fourth vertex D of a tetrahedron ABCD enjoying the property that perpendiculars from the vertices on the opposite faces concur.

(*a*) Show that the point in which the line meets the plane of the triangle ABC is the extremity of the vector,

$$\text{OH}=\eta=\frac{S\alpha\beta\gamma-\alpha S\alpha(\beta-\gamma)-\beta S\beta(\gamma-\alpha)-\gamma S\gamma(\alpha-\beta)}{V(\beta\gamma+\gamma\alpha+\alpha\beta)},$$

and express this vector in the form,

$$\eta=\frac{x\alpha+y\beta+z\gamma}{x+y+z}.$$

(*b*) Show that the line may be represented by

$$\rho=\frac{V\beta\gamma(t-S\beta\gamma)+V\gamma\alpha(t-S\gamma\alpha)+V\alpha\beta(t-S\alpha\beta)}{S\alpha\beta\gamma},$$

Ex. 19. When the vector to a point P in the plane of ABC is expressed in the form,

$$\rho=\frac{x\alpha+y\beta+z\gamma}{x+y+z},$$

show that the ratios of x, y, and z are the ratios of the triangles PBC, PCA, PAB.

(*a*) Hence, if upper and lower signs correspond,

$$\rho=\frac{\alpha T(\beta-\gamma)\pm\beta T(\gamma-\alpha)\pm\nu T(\alpha-\beta)}{T(\beta-\gamma)\pm T(\gamma-\alpha)\pm T(\alpha-\beta)}$$

are the vectors to the centres of the inscribed and escribed circles of the triangle.

(*b*) Deduce the corresponding theorem for a tetrahedron, and find the vectors to the centres of the inscribed and escribed spheres.

Ex. 20. Selecting any point U in the plane of three given points A, B, C, so that

$$\text{OU}=\upsilon=\frac{a\alpha+b\beta+c\gamma}{a+b+c},$$

where a, b, c are constant scalars; the vector to any variable point in the plane may be represented by

$$\text{OP}=\rho=\frac{xa\alpha+yb\beta+zc\gamma}{xa+yb+zc},$$

x, y and z being the anharmonic coordinates of the point P.

(*a*) If $x^2+y^2+z^2-2yz-2zx-2xy=0$, the locus of P is a conic touching the sides of the triangle ABC in points which connect through U to the opposite vertices.

(*b*) If $yz+zx+xy=0$, the locus of P is a conic circumscribing ABC, and the tangents at the vertices intersect the opposite sides in points on the polar of U with respect to the triangle ABC, or with respect to either conic.

(*c*) The two conics have double contact, the polar of U being the chord of contact, and the anharmonic coordinates of the points of contact being 1, ω, ω^2 and 1, ω^2, ω where ω is an algebraic imaginary cube root of unity.

(*d*) Given three scalars, u, v and w, discuss the arrangement of the six points whose anharmonic coordinates are equal to these scalars taken in different orders. Show that the six points lie on a conic. Examine the three cases in which permutation of the scalars determines less than six points.

CHAPTER VI.

THE SPHERE.

ART. **41.** The equation

$$\mathrm{TEP}=\mathrm{T}(\rho-\epsilon)=a, \text{ or } \rho^2-2\mathrm{S}\rho\epsilon+\epsilon^2+a^2=0 \text{ ....(I.)}$$

requires the variable point P to remain at a constant distance a from a fixed point E, and consequently represents a sphere of radius a and of centre E.

The right line $\rho=\beta+t\alpha$ meets the sphere in the points determined by the values of t which satisfy

$$\mathrm{T}(\beta-\epsilon+t\alpha)=a, \text{ or } \mathrm{T}(\beta-\epsilon)^2-a^2-2t\mathrm{S}(\beta-\epsilon)\alpha+t^2\mathrm{T}\alpha^2=0; \text{ (II.)}$$

and the product of the intercepts between the point B and the sphere is independent of α, being

$$t_1t_2\mathrm{T}\alpha^2=\mathrm{T}(\beta-\epsilon)^2-a^2, \text{ ....(III.)}$$

while the sum of the intercepts is

$$(t_1+t_2)\mathrm{T}\alpha=2\mathrm{S}(\beta-\epsilon)\mathrm{U}\alpha, \text{ ....(IV.)}$$

if t_1 and t_2 are the roots of the quadratic (II.).

The square of the chord cut off by the sphere is

$$(t_1-t_2)^2\mathrm{T}\alpha^2=4a^2-4\mathrm{TV}(\beta-\epsilon)\mathrm{U}\alpha^2, \text{ ....(V.)}$$

remembering that $(\mathrm{S}\lambda\mu)^2+\mathrm{T}(\mathrm{V}\lambda\mu)^2=\mathrm{T}\lambda^2\mu^2$ (Art. 17), and accordingly the line meets the sphere in real points, only if

$$\mathrm{TV}(\beta-\epsilon)\mathrm{U}\alpha\leq a, \text{ ....(VI.)}$$

that is, if the perpendicular from the point E on the line is less or equal to the radius of the sphere. For contact,

$$\mathrm{TV}(\beta-\epsilon)\alpha=a\mathrm{T}\alpha; \text{ and } \mathrm{TV}(\beta-\epsilon)(\rho-\beta)=a\mathrm{T}(\rho-\beta)\text{...(VII.)}$$

represents the tangent cone from the point B, BP being a tangent line. Since $\mathrm{TV}\lambda\mu\leq\mathrm{T}\lambda\mathrm{T}\mu$, the cone is real only when $\mathrm{T}(\beta-\epsilon)\geq a$.

The locus of the centres of the chords is derived from (IV.) by putting $\frac{1}{2}(t_1+t_2)\alpha=\rho-\beta$, and is given by

$$\mathrm{S}\frac{\epsilon-\beta}{\rho-\beta}=1, \text{ ....(VIII.)}$$

which represents a sphere on BE as diameter. For it expresses that the projection of BE on BP is equal to BP, so that the angle BPE is right.

Taking the harmonic mean of the vector intercepts to be $\rho-\beta$, we have by (III.) and (IV.),

$$\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\frac{1}{a}=\frac{2}{\rho-\beta}; \text{ and } S(\rho-\epsilon)(\beta-\epsilon)+a^2=0 \text{ ....(IX.)}$$

is the locus of its extremity—the polar plane of the point B.

ART. **42.** Any two spheres,

$$\rho^2-2S\alpha\rho+l=0,\ \ \rho^2-2S\beta\rho+m=0, \text{ ....(I.)}$$

intersect in the plane,

$$2S(\alpha-\beta)\rho=l-m; \text{ ....(II.)}$$

and if P is any point on the second sphere and P′ any point in this radical plane, the power of the first point P with respect to the first sphere is (Art. 41 (III.)),

$$T\rho^2+2S\alpha\rho-l=2S(\alpha-\beta)\rho-l+m=2S(\alpha-\beta)(\rho-\rho'), \text{...(III.)}$$

or twice the projection of PP′ on the line of centres into the distance between the centres.

The spheres cut at an angle determined by

$$\cos\theta=\frac{l+m-2S\alpha\beta}{2\sqrt{\{(T\alpha^2+l)(T\beta^2+m)\}}}, \text{ ....(IV.)}$$

since if a and b are their radii, $a^2+b^2-2ab\cos\theta=T(\alpha-\beta)^2$.

For further investigation, the origin should be taken at the intersection of the line of centres with the radical plane.

A variable sphere cuts two given spheres at constant angles, prove that it cuts an infinite number of spheres at constant angles. Let the sphere (I.), determined by β and m, be the variable sphere, and let it cut the spheres (α, l) and (α', l') at the angles θ and θ'. Assume that it cuts the sphere (α'', l'') at the angle θ''. Then the third of the equations,

$$l+m-2S\alpha\beta=2ab\cos\theta;\ \ l'+m-2S\alpha'\beta=2a'b\cos\theta';$$
$$l''+m-2S\alpha''\beta=2a''b\cos\theta'',$$

analogous to (IV.), must be equivalent to a linear combination of the other two. Multiply by scalars, x, y and z; add and separately equate to zero the coefficients of the variables, m, β and b, and

$$xl+yl'+zl''=0;\ \ x+y+z=0;\ \ x\alpha+y\alpha'+z\alpha''=0;$$
$$xa\cos\theta+ya'\cos\theta'+xa''\cos\theta''=0.$$

The first, second and third show that the sought sphere (a'', l'') must be coaxial with the given spheres, and we have, in fact, on elimination of x, y and z,

$$a''(l-l')+a(l'-l'')+a'(l''-l)=0,$$
$$a''\cos\theta''(l-l')+a\cos\theta(l'-l'')+a'\cos\theta'(l''-l)=0.$$

Substituting for a'' its value, $\sqrt{(\mathrm{T}a''^2+l'')}$, the equation

$$\cos\theta''\sqrt{\{\mathrm{T}[a(l'-l'')+a'(l''-l)]^2+l''(l-l')^2\}^{\frac{1}{2}}}$$
$$+a\cos\theta(l'-l'')+a'\cos\theta'(l''-l)=0$$

becomes a quadratic, which gives two values of l'' for each value of $\cos\theta''$. One sphere only is cut at right angles because the condition becomes linear in l''.

Ex. Reduce the equations of a pair of spheres to the form,

$$\rho^2-2u\mathrm{S}\alpha\rho+l=0\,;\quad \rho^2-2v\mathrm{S}\alpha\rho+l=0,\ \text{where } \mathrm{T}\alpha=1.$$

(a) Prove that all spheres of the family obtained by giving various values to w in

$$\rho^2-2w\mathrm{S}\alpha\rho+l=0$$

intersect in a common circle.

(b) Examine the condition for the reality of the circle, and show that whether real or imaginary, it lies in a real plane.

(c) If the circle is imaginary, there are two real point spheres of the family. Find them.

(d) The spheres of the doubly infinite family

$$\rho^2-2\mathrm{S}\beta\rho-l=0,\quad \mathrm{S}\beta\alpha=0,$$

formed by giving all possible values to the vector β, cut the spheres of the family (a) at right angles.

ART. **43.** Given any three spheres,

$$\rho^2-2\mathrm{S}\alpha\rho+l=0,\ \rho^2-2\mathrm{S}\beta\rho+m=0,\ \rho^2-2\mathrm{S}\gamma\rho+n=0\,;\ldots(\text{I.})$$

the radical planes of each pair intersect in the line,

$$2\mathrm{S}\alpha\rho-l=2\mathrm{S}\beta\rho-m=2\mathrm{S}\gamma\rho-n\,;\ \ldots\ldots\ldots\ldots\ldots(\text{II.})$$

or $\rho=\frac{1}{2}(l\mathrm{V}\beta\gamma+m\mathrm{V}\gamma\alpha+n\mathrm{V}\alpha\beta)(\mathrm{S}\alpha\beta\gamma)^{-1}+t\mathrm{V}(\beta\gamma+\gamma\alpha+\alpha\beta).(\text{III.})$

If the origin is taken on this line, $l=m=n$; and if it is taken where the line intersects the plane of centres ABC, the equations of the spheres may be reduced to the type,

$$\rho^2-2\mathrm{S}\kappa\rho+l=0,\ \mathrm{S}\kappa\nu=0,\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{IV.})$$

the vector ν being fixed, but κ being susceptible of various values.

The spheres of this family (IV.) of given radius (a) have their centres on a fixed circle,

$$\mathrm{T}\kappa=\sqrt{(a^2-l)},\ \mathrm{S}\kappa\nu=0.$$

It is easy to verify that the radical axes of every three out of four given spheres intersect in a point. This point is the radical centre of the four spheres, and is situated at the extremity of the vector,

$$\rho = \tfrac{1}{2}\Sigma \frac{l\mathrm{V}(\beta\gamma+\gamma\delta+\delta\beta)}{\mathrm{S}\alpha\mathrm{V}(\beta\gamma+\gamma\delta+\delta\beta)}, \qquad \text{(v.)}$$

the fourth sphere being $\rho^2 - 2\mathrm{S}\delta\rho + p = 0$.

It may be verified that if in this equation p and δ are rendered arbitrarily variable, we fall back on the radical axis of three spheres. If, in addition, γ and n are arbitrary, the same equation represents the radical plane of two. For example, we may put $\delta = x\alpha + y\beta + z\gamma$, where x, y and z are arbitrary.

Ex. 1. Find the locus of the centre of a sphere cutting three spheres orthogonally.

[Let δ and p determine the sphere whose centre is sought, and let the three spheres belong to the family (IV.). The condition $l + p - 2\mathrm{S}\delta\kappa = 0$ must be satisfied by three values of the vector κ. Hence $p = -l$, $\delta \parallel \nu$, and the locus is the radical axis.]

Ex. 2. Find a sphere cutting four spheres orthogonally.

Ex. 3. If four spheres are mutually orthogonal, their centres determine a tetrahedron self-conjugate to a sphere.

[Let the spheres be referred to their radical centre. The conditions are $l = \mathrm{S}\alpha\beta = \mathrm{S}\alpha\gamma = \mathrm{S}\alpha\delta = \mathrm{S}\beta\gamma = \mathrm{S}\beta\delta = \mathrm{S}\gamma\delta$, and the centres are conjugate in pairs to the sphere $\rho^2 = l$.]

The Method of Inversion.

ART. **44.** We have seen that

$$\rho^{-1} = \mathrm{OP}^{-1} = \mathrm{OP}' = \rho'$$

represents a vector having its tensor reciprocal and its direction opposite to the tensor and the direction of the vector ρ (Art. 16). Hence more generally if

$$\mathrm{CP}' = \rho' - \gamma = -R^2(\rho-\gamma)^{-1} = -R^2 \,.\, \mathrm{CP}^{-1}, \qquad \text{(I.)}$$

P and P′ are inverse points with respect to the sphere, centre C and radius R, for

$$\mathrm{UCP}' = \mathrm{UCP}, \quad \mathrm{TCP}'\mathrm{TCP} = R^2.$$

The inverse of the sphere $\mathrm{T}(\rho - \alpha) = a$ is

$$\mathrm{T}\left(\gamma - \alpha - \frac{R^2}{\rho-\gamma}\right) = a, \;\text{ or }\; \mathrm{T}(\alpha-\gamma)^2 - 2R^2\mathrm{S}\frac{\alpha-\gamma}{\rho-\gamma} + \frac{R^4}{\mathrm{T}(\rho-\gamma)^2} = a^2;$$

or

$$\mathrm{T}\left(\rho - \gamma - \frac{(\alpha-\gamma)R^2}{\mathrm{T}(\alpha-\gamma)^2 - a^2}\right) = \mathrm{T}\frac{aR^2}{\mathrm{T}(\alpha-\gamma)^2 - a^2}. \qquad \text{(II.)}$$

The symbol T prefixed to the scalar on the right is intended to show that it is to be taken positively. Thus, to invert the given sphere into a sphere of radius b, we have

$$b = \pm\frac{aR^2}{\mathrm{T}(\alpha-\gamma)^2 - a^2} \text{ according as } \mathrm{T}(\alpha-\gamma) > \text{ or } < a, \qquad \text{(III.)}$$

or according as the centre of inversion lies outside or inside the given sphere.

The inverse of a plane is a sphere through the centre of inversion, and the inverse of a line is a circle. Thus

$$\gamma-\frac{R^2}{\rho-\gamma}=\alpha+t\beta, \text{ or } \mathrm{V}\left(\frac{R^2}{\rho-\gamma}+\alpha-\gamma\right)\beta=0, \ldots\ldots\ldots \text{(IV.)}$$

represents a circle through the point C—the inverse of the line $\rho=\alpha+t\beta$.

Ex. 1. If any two vectors OA, OB have OA′, OB′ for their reciprocals, then the right line A′B′ is parallel to the tangent OD at the origin O, to the circle OAB; and the two triangles, OAB, OB′A′, are inversely similar. (*Elements of Quaternions*, Art. 259.)

Ex. 2. Invert the sphere, centre A and radius a, into the sphere centre B and radius b.

[Here $$\beta=\gamma+\frac{(\alpha-\gamma)R^2}{\mathrm{T}(\alpha-\gamma)^2-a^2}, \quad \frac{aR^2}{\mathrm{T}(\alpha-\gamma)^2-a^2}=\pm b,$$

and from these

$$\gamma=\frac{\pm ab-\beta a}{\pm b-a}, \quad \text{and} \quad R^2=\frac{\pm ab}{(\pm b-a)^2}(\mathrm{T}(\alpha-\beta)^2-(\pm b-a)^2).$$

There are two real positions for the centre, but there may be only one *positive* value of R^2.]

Ex. 3. Invert a system of coaxial spheres into concentric spheres.

[A system of coaxial spheres $\rho^2-2w\mathrm{S}\alpha\rho+l=0$ inverts into a system of spheres having their centres on the line locus,

$$\beta=\gamma-\frac{(w\alpha-\gamma)R^2}{\gamma^2-2w\mathrm{S}\gamma\alpha+l}.$$

If this is independent of w, it is easy to see that $\gamma^2-l=0$, $\gamma \parallel \alpha$, or $\gamma=\pm\alpha\sqrt{-l}$.

The centre of the inverted spheres is $\pm\alpha\sqrt{-l}\mp\frac{1}{2}\alpha R^2 : \sqrt{-l}$.

Ex. 4. Prove that

$$\rho=\frac{\dfrac{x\alpha}{\alpha-\delta}+\dfrac{y\beta}{\beta-\delta}+\dfrac{z\gamma}{\gamma-\delta}}{\dfrac{x}{\alpha-\delta}+\dfrac{y}{\beta-\delta}+\dfrac{z}{\gamma-\delta}}$$

represents a sphere through the four points A, B, C and D.

[Invert with respect to the point D.]

ART. **45.** The following examples relating to a sphere and a tetrahedron are easily solved by the formulae X. or XII. of Art. 38, or by the method of Art. 39.

Ex. 1. Determine the sphere through A, B, C and D.

[The vector κ to the centre is $\kappa=-\frac{1}{2}v^{-1}\Sigma\lambda\alpha^2=-\frac{1}{2}\Sigma\lambda_{\prime}^{-1}\alpha^2$, and the squared radius is $R^2=-v^{-1}\Sigma l\alpha^2-\frac{1}{4}v^{-2}(\Sigma\lambda\alpha^2)^2$.]

Ex. 2. Given four spheres having their centres at A, B, C and D, and their radii equal to a, b, c, d, find their radical centre.

[If ω is the vector to the radical centre, and if $h=(\omega-\alpha)^2+a^2$, we have

$$\omega=-\tfrac{1}{2}v^{-1}\Sigma\lambda(\alpha^2+a^2), \quad h=v^{-1}\Sigma l(\alpha^2+a^2)+\tfrac{1}{4}v^{-2}(\Sigma\lambda(\alpha^2+a^2))^2.]$$

Ex. 3. Describe a sphere to cut four spheres orthogonally.

Ex. 4. Describe a sphere to cut four given spheres at given angles.

[Here there are four equations of the form $(\kappa-\alpha)^2+a^2-2aR\cos\theta+R^2=0$. Multiplying by the scalars l and the vectors λ and forming the sums, the equations,

$$v(\kappa^2+R^2)+\Sigma l(a^2+\alpha^2)-2R\Sigma la\cos\theta=0\;;\quad 2\kappa v+\Sigma\lambda(a^2+\alpha^2)-2R\Sigma\lambda a\cos\theta=0,$$

are obtained. Substitution for κ in the first gives a quadratic in R. For the origin at the radical centre, the equations are,

$$R^2\{(\Sigma\lambda a\cos\theta)^2+v^2\}-2Rv\Sigma la\cos\theta+hv^2=0\;;\quad \kappa v=R\Sigma\lambda a\cos\theta.]$$

Ex. 5. To invert four spheres into four others of given radii.

[If a', b', c', d' are the radii which the inverted spheres are required to have, and if the vector ι terminates at the centre of inversion,

$$\iota^2-2S\iota\alpha+\alpha^2+a^2\pm\frac{a}{a'}R^2=0.\quad \text{(Ex. 2 of last Article.)}$$

Taking the origin at the radical centre,

$$v(\iota^2+h)+R^2\Sigma\pm\frac{a}{a'}l=0,\quad 2v\iota+R^2\Sigma\pm\frac{a}{a'}\lambda=0.$$

These lead to a quadratic in R^2 for each set of signs.]

Ex. 6. Find the equation of a sphere touching the four faces of a tetrahedron.

$$[0=v+r\Sigma\pm T\lambda\;;\quad 0=v\kappa+r\Sigma\pm\alpha T\lambda.]$$

Ex. 7. Find the condition that five points A, B, C, D, E should lie on a sphere.

[In the notation of Art. 39, p. 43, this is $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\epsilon^2=0$, or

$$\text{OA}^2(\text{BCDE})-\text{OB}^2(\text{ACDE})+\text{OC}^2(\text{ABDE})-\text{OD}^2(\text{ABCE})+\text{OE}^2(\text{ABCD})=0.]$$

Ex. 8. If five spheres are orthogonal to a sphere, prove that

$$\text{P}_\text{A}(\text{BCDE})-\text{P}_\text{B}(\text{ACDE})+\text{P}_\text{C}(\text{ABDE})-\text{P}_\text{D}(\text{ABCE})+\text{P}_\text{E}(\text{ABCD})=0,$$

where A, B, C, D, E are centres of the spheres and where P_A, P_B, P_C, P_D, and P_E are the powers of any point with respect to the five spheres.

Ex. 9. If five spheres cut a sixth at the angles θ, θ', etc., prove that the radius (R) of the sixth is given by the relation

$$\Sigma\text{P}_\text{A}(\text{BCDE})=2R\Sigma a\cos\theta(\text{BCDE}),$$

P_A being defined as in the last Example, and a, b, c, d, e being the radii of the five spheres.

Ex. 10. Find the equation of a sphere in anharmonic coordinates.

[Compare Art. 40, p. 43. The imaginary cone standing on the circle at infinity is

$$T\rho^2=0,\quad\text{or}\quad \Omega=\Sigma T\alpha^2a^2x^2-2\Sigma S\alpha\beta abxy=0,$$

and a sphere is

$$\Omega+\Sigma ax\Sigma lx=0.]$$

Ex. 11. Prove that the equation of the sphere circumscribing the tetrahedron ABCD is in anharmonic coordinates,

$$\Sigma T(\alpha-\beta)^2abxy=0.$$

ART. **46.** *The product of the successive vector sides of a polygon of odd order inscribed in a sphere is a tangential vector at the initial point of the polygon; and if the number of sides is even, the product is a quaternion whose vector part is parallel to the vector radius to the initial point.*

The centre of the sphere being O, and A_1, A_2 being successive vertices, the isosceles triangle A_2A_1O is inversely similar to A_1A_2O, and therefore (Art. 18, p. 14),

$$\frac{A_1O}{A_1A_2} = K\frac{A_2O}{A_2A_1} = \frac{1}{A_2A_1} \cdot A_2O, \text{ or } OA_2 = -A_1A_2 \cdot OA_1 \cdot \frac{1}{A_1A_2}.$$

Thus, if $OA_1 = \alpha_1$, $OA_2 = \alpha_2$, etc., $A_1A_2 = \gamma_1$, $A_2A_3 = \gamma_2$, etc.,

$$\alpha_2 = -\gamma_1\alpha_1\gamma_1^{-1},\ \alpha_3 = -\gamma_2\alpha_2\gamma_2^{-1} = +\gamma_2\gamma_1\alpha_1\gamma_1^{-1}\gamma_2^{-1}, \text{ etc.};$$

and generally, the polygon being closed so that $\alpha_{n+1} = \alpha_1$,

$$\alpha_1 = (-)^n q\alpha_1 q^{-1}, \text{ where } q = \gamma_n\gamma_{n-1} \dots \gamma_2\gamma_1. \quad \text{(I.)}$$

For an odd number of sides,

$$q\alpha_1 + \alpha_1 q = 0, \text{ or } \alpha_1 Sq + S\alpha_1 Vq = 0, \text{ or } Sq = 0,\ Vq \perp \alpha_1; \quad \text{(II.)}$$

and for an even number,

$$q\alpha_1 - \alpha_1 q = 0, \text{ or } V \cdot \alpha_1 Vq = 0, \text{ or } Vq \parallel \alpha_1. \quad \text{(III.)}$$

In the first case (n odd), the product is a vector, and is perpendicular to α_1, or parallel to a tangent at A_1. In the second case (n even), the product is a quaternion having its vector part parallel to α_1.

In connection with this article and its examples, Art. 296 of the *Elements of Quaternions* should be consulted.

Ex. 1. The equation of the sphere through four given points A, B, C, D may be written in the form

$$S(\rho-\alpha)(\alpha-\beta)(\beta-\gamma)(\gamma-\delta)(\delta-\rho) = 0.$$

Ex. 2. The normal at the point P on this sphere is parallel to

$$V(\rho-\alpha)(\alpha-\beta)(\beta-\gamma)(\gamma-\rho);$$

and the vector ϖ being variable,

$$S(\varpi-\rho)(\rho-\alpha)(\alpha-\beta)(\beta-\gamma)(\gamma-\rho) = 0$$

is the equation of the tangent plane at P.

Ex. 3. The equation of the circle ABC is

$$V(\rho-\alpha)(\alpha-\beta)(\beta-\gamma)(\gamma-\rho) = 0,$$

and the tangent to the circle at the point P is

$$V(\varpi-\rho)(\rho-\alpha)(\alpha-\beta)(\beta-\rho) = 0.$$

[The vector part of a product of an even number of coplanar vectors is perpendicular to their plane, being a product of half the number of coplanar quaternions. Therefore when the points are coplanar the expression for the normal vector in Ex. 2 must vanish, as this vector cannot be perpendicular to the plane. The equation is also susceptible of geometrical interpretation.]

Ex. 4. The product of four successive vector sides of a quadrilateral inscribed to a circle is a positive or negative scalar according as the quadrilateral is crossed or uncrossed.

[Use the relation $\mathrm{U}.\frac{\mathrm{AB}}{\mathrm{BC}} = \pm\mathrm{U}\frac{\mathrm{AD}}{\mathrm{DC}}$, which asserts that the angles ABC, ADC are equal or supplementary.]

Ex. 5. The "anharmonic function of four points in space" being defined by the equation

$$(\mathrm{ABCD}) = \frac{\mathrm{AB}}{\mathrm{BC}} \cdot \frac{\mathrm{CD}}{\mathrm{DA}},$$

examine the nature of this quaternion when the four points are concyclic.

Ex. 6. Prove that the anharmonic functions of any four points in space satisfy the relations

$$(\mathrm{ABCD}) + (\mathrm{ACBD}) = 1, \quad (\mathrm{ABCD}).(\mathrm{ADCB}) = 1\,;$$

and that

$$(\mathrm{ABCD}) = \mathrm{K}\frac{\mathrm{C'D'}}{\mathrm{C'B'}},$$

where B′, C′ and D′ are the inverse points of B, C and D with respect to the point A.

[Note that $\alpha^{-1} - \beta^{-1} = \alpha^{-1}.(\beta - \alpha)\beta^{-1}$.]

Ex. 7. If $(\mathrm{OABC}) = -1$, prove that $\mathrm{OB}^{-1} = \frac{1}{2}(\mathrm{OA}^{-1} + \mathrm{OC}^{-1})$.

Ex. 8. Inscribe a polygon to a sphere, given the directions of the sides of the polygon.

[Here $\mathrm{U}q$ is given, q denoting the quaternion in the text; and (II.) and (III.) show that the vector to the first corner is $\perp \mathrm{VU}q$, or else $\parallel \pm\mathrm{VU}q$.]

Ex. 9. For the gauche quadrilateral OABC, which may always be conceived to be inscribed in a determined sphere, we may say that the angle of the quaternion product, $\angle(\mathrm{OA}.\mathrm{AB}.\mathrm{BC}.\mathrm{CO})$, is equal to the angle of the lunule, bounded by the two arcs of small circles OAB, OCB; with the same construction for the angle of the anharmonic $\angle(\mathrm{OABC})$, or $\angle(\mathrm{OA} : \mathrm{AB}.\mathrm{BC} : \mathrm{CO})$. (*Elements*, Art. 296 (15).)

Ex. 10. Let ABCD be any four points in a plane or in space, connected by four circles, each passing through three of the points; then, not only is the angle at A, between the arcs ABC, ADC, equal to the angle at C, between CDA and CBA, but also it is equal to the angle at B, between the two other arcs BCD and BAD, and to the angle at D, between the arcs DAB, DCB. (*Elements*, Art. 296 (18).)

Ex. 11. The vector part of the product of four successive sides of a gauche quadrilateral inscribed in a sphere is equal to the diameter drawn to the initial point of the polygon, multiplied by the sextuple volume of the pyramid, which its four points determine. (*Elements*, Art. 296 (43).)

ART. **47**. *To inscribe a polygon in a sphere so that its sides may pass through given points.*

Let the unit of length be selected equal to the radius of the sphere. Let the centre be taken as origin, and let $\rho, \rho_1, \rho_2, \ldots \rho_n (= \rho)$ be the vectors to the vertices, while $\beta_1, \beta_2, \ldots \beta_n$ are the vectors to the fixed points. The rectangle under the segments of the chords through β_1 is

$$(\rho-\beta_1)(\rho_1-\beta_1)=1+\beta_1^2; \qquad \text{(I.)}$$

so that $$\rho_1=-\frac{\beta_1\rho+1}{\beta_1-\rho}=-\frac{p_1\rho+q_1}{p_1-q_1\rho} \text{ if } p_1=\beta_1,\ q_1=1. \qquad \text{(II.)}$$

Again,

$$\rho=-\frac{\beta_2\rho_1+1}{\beta_2-\rho_1}=(-)^2\frac{p_2\rho+q_2}{p_2-q_2\rho} \text{ if } p_2=\beta_2p_1+q_1,\ q_2=\beta_2q_1-p_1; \qquad \text{(III.)}$$

and it is easy to see that, in general,

$$\rho_m=(-)^m\frac{p_m\rho+q_m}{p_m-q_m\rho} \text{ if } \left.\begin{aligned} p_m&=\beta_mp_{m-1}-(-)^{m-1}q_{m-1},\\ q_m&=\beta_mq_{m-1}+(-)^{m-1}p_{m-1}.\end{aligned}\right\} \qquad \text{(IV.)}$$

Finally, $$\rho=(-)^n\frac{p\rho+q}{p-q\rho} \text{ if } \rho_n=\rho,\ p_n=p,\ q_n=q. \qquad \text{(V.)}$$

Two cases now arise according as n is odd or even. In the first place, if n is odd, remembering that $\rho^2=-1$,

$$\rho p+p\rho=\rho q\rho-q=\rho(q\rho+\rho q); \text{ or } \rho Sp+S\rho p=\rho(\rho Sq+S\rho q);$$

or, separating the scalar and the vector parts,

$$S\rho p+Sq=0 \text{ and } S\rho q-Sp=0. \qquad \text{(VI.)}$$

Introducing the imaginary of algebra, these may be combined into the single relation,

$$S(\rho+\sqrt{-1})(q+\sqrt{-1}p)=0. \qquad \text{(VII.)}$$

The equations (VI.) give a line locus for ρ which intersects the sphere in two points—real or imaginary—which satisfy the conditions.

In the second place, if n is even,

$$\rho p-p\rho=\rho q\rho+q=\rho(q\rho-\rho q); \text{ or } V.\rho Vp=\rho V.Vq\rho.$$

Adding to each side $x=S\rho p$, we have

$$V\rho Vq+Vp=\rho^{-1}x=-x\rho; \text{ and this gives } SVpVq=-xS\rho Vq$$

on operating by SVq. Hence,

$$\rho(Vq+x)=-Vp-x^{-1}SVpVq,$$

as we see by adding $S\rho Vq$ to each side. Thus,

$$\rho=-\frac{xVp+SVpVq}{x(Vq+x)} \text{ and } x^4+x^2(TVq^2-TVp^2)-(SVpVq)^2=0, \qquad \text{(VIII.)}$$

as appears on taking the tensor, remembering that $T\rho^2=1$. This quadratic in x^2 has one negative root. The other root is positive, and there are thus two real values for x, and two real points satisfying the conditions.

We have now to determine p and q. Multiply p_m in equation (IV.) by $\sqrt{-1}$ and add it to q_m, and

$$q_m+\sqrt{-1}p_m=(\beta_m-(-)^{m-1}\sqrt{-1})(q_{m-1}+\sqrt{-1}p_{m-1})$$
$$=(\beta_m+(-)^m\sqrt{-1})(q_{m-1}+\sqrt{-1}p_{m-1}).$$

This gives at once, on referring to (II.),

$$\left.\begin{array}{l} q+\sqrt{-1}p=(\beta_n+(-)^n\sqrt{-1})(\beta_{n-1}+(-)^{n-1}\sqrt{-1}) \\ \quad\dots(\beta_2+(-)^2\sqrt{-1})(\beta_1-\sqrt{-1})\sqrt{-1} \end{array}\right\}; \quad\dots(\text{IX.})$$

and the real and imaginary parts of this product are q and p.

A quaternion of the form $q+\sqrt{-1}\,.\,p$ is called by Hamilton a bi-quaternion. (Compare Art. 22, p. 20.)

Ex. Show that in the notation of this article

$$Tq^2-Tp^2=(-)^{n+1}(\beta_n{}^2+1)(\beta_{n-1}{}^2+1)\dots(\beta_1{}^2+1); \quad SqKp=0.$$

[Multiply $q+\sqrt{-1}p$ into $Kq+\sqrt{-1}Kp$ and separate the real and the imaginary parts.]

EXAMPLES TO CHAPTER VI.

Ex. 1. The sphere which has its centre at the origin, and has the vector OA, or α, with a length $T\alpha=a$, for one of its radii, may be represented by any one of the following equations :

$$\frac{\alpha}{\rho}=K\frac{\rho}{\alpha}; \quad S\frac{\rho-\alpha}{\rho+\alpha}=0; \quad S\frac{2\alpha}{\rho+\alpha}=1; \quad S\frac{2\rho}{\rho+\alpha}=1; \quad T\left(S\frac{\rho}{\alpha}+V\frac{\rho}{\alpha}\right)=1;$$
$$T(\rho-c\alpha)=T(c\rho-\alpha),$$

which are transformations one of the other, and each of which exhibits some geometrical property of the surface.

Ex. 2. The circle which has its centre at the origin, which lies in the plane $S\alpha\rho=0$, and which has $T\alpha$ for its radius, is represented by the equation

$$\left(\frac{\rho}{\alpha}\right)^2=-1.$$

Ex. 3. If t is a variable parameter, in absolute magnitude not greater than unity, the equations

$$S\frac{\rho}{\alpha}=t, \quad \left(V\frac{\rho}{\alpha}\right)^2=t^2-1,$$

represent a system of circles which generate a sphere.

Ex. 4. The equation of the sphere through the four points O, A, B, C may be written in the forms

$$S(OA\,.\,AB\,.\,BC\,.\,CP\,.\,PO)=0;$$
$$\alpha^2S\beta\gamma\rho+\beta^2S\gamma\alpha\rho+\gamma^2S\alpha\beta\rho=\rho^2S\alpha\beta\gamma;$$
$$S(\beta^{-1}-\alpha^{-1})(\gamma^{-1}-\alpha^{-1})(\rho^{-1}-\alpha^{-1})=0.$$

Ex. 5. If we project the variable point P of a sphere into points A$'$, B$'$, C$'$ on the three given chords OA, OB, OC by three planes through that point P parallel to the planes BOC, COA, AOB, we shall have the equation

$$OP^2=OA\,.\,OA'+OB\,.\,OB'+OC\,.\,OC'$$

Ex. 6. The expression*

$$\rho=rk^t j^s k j^{-s} k^{-t} \quad \text{or} \quad \rho=rk^t j^{2s} k^{1-t},$$

in which r is a given scalar, i, j, k mutually rectangular unit vectors, while s and t are parameters, represents a sphere concentric with the origin.

The expression may also be put under the form

$$\rho=r\mathrm{V}\,.\,k^{2s+1}+rk^{2t}\mathrm{V}\,.\,i^{2s},$$

and it may be expanded as follows:

$$\rho=r\{(i\cos t\pi+j\sin t\pi)\sin s\pi+k\cos s\pi\}.$$

(a) Show how to establish the first form of the expression by the properties of conical rotations.

Ex. 7. Show that the equation

$$\left(\frac{w+\rho-\alpha}{\beta}\right)^2=-1,$$

in which w is a real scalar capable of receiving any value consistent with the reality of the vector ρ, represents the portion of the plane $\mathrm{S}(\rho-\alpha)\beta=0$ included within the sphere $\mathrm{T}(\rho-\alpha)=\mathrm{T}\beta$.

Ex. 8. The equation† $\quad \mathrm{T}(w+\rho)=1,$

in which ρ is a real variable vector and w a real variable scalar, represents the region enclosed by the sphere $\mathrm{T}\rho=1$.

Ex. 9. A sphere passes through the intersection of the planes $\mathrm{S}\lambda\rho=0$, $\mathrm{S}\mu\rho=0$, $\mathrm{S}\nu\rho=0$, which cut off caps the sum of whose areas is equal to $2\pi a^2$. Show that the locus of the centre is represented by

$$3\mathrm{T}\rho^2+\mathrm{T}\rho\,.\,\mathrm{S}(\mathrm{U}\lambda+\mathrm{U}\mu+\mathrm{U}\nu)\rho=a^2.$$

Ex. 10. The centre of a sphere of constant radius a describes a circle of radius b concentric with the origin and in the plane $\mathrm{S}\alpha\rho=0$, $\mathrm{T}\alpha=1$. The equation of the surface generated may be written

$$\mathrm{T}(\pm b\mathrm{U}\,.\,\alpha^{-1}\mathrm{V}\alpha\rho-\rho)=a\,;$$

or
$$2b\mathrm{TV}\alpha\rho=\pm(\mathrm{T}\rho^2+b^2-a^2)\,;$$

or
$$4b^2(\mathrm{S}\alpha\rho)^2=4b^2\mathrm{T}\rho^2-(\mathrm{T}\rho^2+b^2-a^2)^2\,;$$

or
$$4a^2\mathrm{T}\rho^2-4b^2(\mathrm{S}\alpha\rho)^2=(\mathrm{T}\rho^2-b^2+a^2)^2\,;$$

or
$$\mathrm{SU}\,.\,\frac{\rho-\alpha(a^2-b^2)^{\frac{1}{2}}}{\rho+\alpha(a^2-b^2)^{\frac{1}{2}}}=\pm\frac{b}{a}\,;$$

or
$$\rho=\pm b\mathrm{U}\,.\,\alpha^{-1}\mathrm{V}\alpha\tau+a\mathrm{U}\tau \quad (\tau \text{ a variable vector}).$$

(a) Taking β and γ, two auxiliary unit-vectors perpendicular to one another and to α, show that

$$a^2\mathrm{T}\rho^2-b^2(\mathrm{S}\alpha\rho)^2=a^2(\mathrm{S}\gamma\rho)^2+\mathrm{S}\rho(a\beta+\alpha\sqrt{b^2-a^2})\mathrm{S}\rho(a\beta-\alpha\sqrt{b^2-a^2}),$$

and prove that each of the planes

$$\mathrm{S}\rho(a\beta\pm\alpha\sqrt{b^2-a^2})=0$$

touches the surface in two points and cuts it in a pair of circles.

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* Examples 1-6 are taken from Hamilton's *Elements of Quaternions*.

† This and the last example are to be found in Hamilton's *Lectures on Quaternions*, Art. 679.

Ex. 11. If p and q are variable quaternions, while α and β are given vectors, show that

$$\text{OP}=\rho=p\alpha p^{-1}+q\beta q^{-1}$$

represents the shell included between the spheres

$$T\rho=T\alpha+T\beta,\quad T\rho=T(T\alpha-T\beta).$$

(*a*) If γ is a third given vector, and if a and b are given scalars, the point P terminates on the circle of intersection of the spheres

$$T(a\rho-\gamma)=T(a-b)\beta,\quad T(b\rho-\gamma)=T(a-b)\alpha,$$

when the quaternions p and q are connected by the relation

$$ap\alpha p^{-1}+bq\beta q^{-1}=\gamma.$$

(*b*) When the relation

$$V\gamma(ap\alpha p^{-1}+bq\beta q^{-1})=0$$

connects p and q, the locus of P is the surface

$$4(S\rho\gamma)^2\{abT\rho^2+(a-b)(aT\alpha^2-bT\beta^2)\}=T\gamma^2\{(a+b)T\rho^2+(a-b)(T\alpha^2-T\beta^2)\}^2.$$

(*c*) If the condition

$$S\gamma(ap\alpha p^{-1}+bq\beta q^{-1})=0$$

is satisfied, the point P must render the expression

$$4(S\rho\gamma)^2\{abT\rho^2+(a-b)(aT\alpha^2-bT\beta^2)\}$$
$$+(a-b)^2T\gamma^2(T\rho^4+T\alpha^4+T\beta^4-2T\alpha^2\beta^2-2T\beta^2\rho^2-2T\rho^2\alpha^2),$$

less than zero.

Ex. 12. The bars AB, BC and CD are connected by universal joints at B and C, and also to two fixed points A and D. If P is a point fixed in BC, and if we write

$$\rho=\text{AP}=\text{AB}+u\text{BC},\quad \rho'=\text{PD}=u'\text{BC}+\text{CD},\quad u+u'=1,$$

where u is a given scalar, and also

$$\text{AB}=p\alpha p^{-1},\quad \text{BC}=q\beta q^{-1},\quad \text{CD}=r\gamma r^{-1},\quad \text{DA}=\delta,$$

where α, β, γ, δ are given vectors and p, q and r variable quaternions, prove that

$$q\beta q^{-1}=\frac{\rho'u'(\rho^2+u^2\beta^2-\alpha^2)-\rho u(\rho'^2+u'^2\beta^2-\gamma^2)+t}{2uu'V\rho\rho'},$$

t being a scalar, and hence show that the inequality

$$T\beta\geq\frac{T\{\rho'u'(\rho^2+u^2\beta^2-\alpha^2)-\rho u(\rho'^2+u'^2\beta^2-\gamma^2)\}}{2Tuu'V\rho\rho'}$$

determines the region within which the point P must lie.

(*a*) If the bar BC remains parallel to the fixed vector β, the locus of P is the intersection of the spheres

$$(\rho-u\beta)^2=\alpha^2,\quad (\rho'-u'\beta)^2=\gamma^2.$$

(*b*) In this case the locus of the bar BC is the cylinder

$$\left[-\delta-\beta+\beta V\frac{\rho}{\beta}\pm\beta\sqrt{\left\{\frac{\alpha^2}{\beta^2}+\left(V\frac{\rho}{\beta}\right)^2\right\}}\right]^2=\gamma^2.$$

(*c*) When the quadrilateral ABCD is coplanar and when the motion is confined to the plane ABCD, find equations of the form

$$\rho=\iota^x\alpha+\iota^y\epsilon,\quad f(x,y)=0,$$

for the path of any point of a plane lamina attached to BC, ι being a constant unit-vector perpendicular to the plane ABCD, and $f(x, y)$ being a scalar function of x and y.

Ex. 13. Solve the equation

$$\frac{1}{\rho-\alpha}+\frac{1}{\rho-\beta}-\frac{1}{\rho-\gamma}-\frac{1}{\rho-\delta}=0.$$

(a) If ρ', α', β' and γ' are the vectors from the point D, the extremity of the vector δ, to the inverses of the extremities of ρ, α, β and γ with respect to D,

$$\frac{1}{\rho'-\alpha'}+\frac{1}{\rho'-\beta'}-\frac{1}{\rho'-\gamma'}=0.$$

Hence deduce the relations

$$\frac{\rho'-\beta'}{\rho'-\alpha'}=\frac{\gamma'-\beta'}{\rho'-\gamma'}=\frac{\rho'-\gamma'}{\gamma'-\alpha'}=\left(\frac{\gamma'-\beta'}{\gamma'-\alpha'}\right)^{\frac{1}{2}}.$$

(b) Solve similarly the quaternion equation

$$\frac{1}{q-a}+\frac{1}{q-b}-\frac{1}{q-c}-\frac{1}{q-d}=0$$

by assuming

$$(q-d)(q'-d)=(a-d)(a'-d)=(b-d)(b'-d)=(c-d)(c'-d)=1.$$

(Robert Russell.)

CHAPTER VII.

DIFFERENTIATION.

ART. **48.** The equation

$$OP = \rho = \phi(t), \quad \text{(I.)}$$

in which a variable vector ρ is given as a function of a variable scalar t, represents a curve in space, it being possible in general to pass from one point P to another point P′ on the locus, only in one definite way—namely, through the series of points determined by the variation of the parameter from t to t'.

The chord PP′ of the curve is

$$PP' = \rho' - \rho = \phi(t') - \phi(t), \quad \text{(II.)}$$

and for the sake of argument we shall suppose that the parameter t represents the time, so that P is the position of a moving point at the time t, and P′ its position at the time t'.

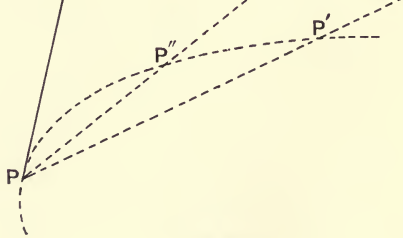

FIG. 23.

Writing $$PQ' = \frac{PP'}{t'-t} = \frac{\phi(t') - \phi(t)}{t'-t}, \quad \text{(III.)}$$

it is apparent that had the point passed from P to P′, in the time

$t'-t$, not along the curve and with varying velocity, but along the chord and with uniform velocity, and that had it continued to move uniformly along the production of the chord, it would have reached the point Q′ in unit time. In a similar manner the point Q″ would have been reached in unit time had the point moved uniformly along the chord PP″ in the time in which it had described the curve and had its motion been continued along the chord without alteration. In the limit PQ represents rigorously the velocity at the point P, in magnitude and direction, for Q is the position the point would have reached in unit time had it left the curve at the point P, preserving unchanged the velocity it actually possessed at that point. The equations

$$\left.\begin{aligned}\mathrm{PQ} &= \lim_{t'=t}\frac{\phi(t')-\phi(t)}{t'-t} = \lim_{h=0}\frac{\phi(t+h)-\phi(t)}{h}\\ &= \lim_{n=\infty} n\left(\phi\left(t+\frac{1}{n}\right)-\phi(t)\right)\end{aligned}\right\}\quad\ldots\ldots\ldots(\text{IV.})$$

are equivalent modes of expressing the limit to which we advance; the third being perhaps in closest agreement with the illustration. It is usual to write

$$\mathrm{PQ} = \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = \phi'(t) \quad\ldots\ldots\ldots\ldots\ldots\ldots(\text{V.})$$

as an abbreviation for the limit.

The vector $\phi'(t)$ is the derivative, the derived or the differential coefficient of the vector function $\phi(t)$ of the *scalar* t, and the *differential* of $\phi(t)$ corresponding to *any* scalar differential $\mathrm{d}t$ of t is

$$\mathrm{d}\,.\,\phi(t) = \lim_{n=\infty} n\left(\phi\left(t+\frac{\mathrm{d}t}{n}\right)-\phi(t)\right) = \phi'(t)\,.\,\mathrm{d}t. \quad\ldots\ldots\ldots(\text{VI.})$$

This is a vector tangential to the curve and of length proportional to the differential $\mathrm{d}t$ which may be large or small.

If t is the arc of the curve, the vector $\phi'(t)$ is of unit length, for in this case we may consider t to represent the time for *unit* and uniform velocity along the curve.

If $\phi'(t)=0$, the extremity of the vector $\mathrm{OP}=\phi(t)$ is a cusp or *stationary* point.

Ex. 1. The curve $\rho=\alpha\cos t+\beta\sin t$

represents an ellipse of which α and β are conjugate radii.

[The vector $\rho'=\dfrac{\mathrm{d}\rho}{\mathrm{d}t}=-\alpha\sin t+\beta\cos t=\alpha\cos\left(\dfrac{\pi}{2}+t\right)+\beta\sin\left(\dfrac{\pi}{2}+t\right)$ is the radius conjugate to ρ.]

Ex. 2. The parallelogram determined by conjugate radii of an ellipse is constant in area.

[$\mathrm{V}\rho\rho'=\mathrm{V}\alpha\beta$.]

Ex. 3. How is the point at the extremity of the vector

$$\alpha\frac{\cos\frac{1}{2}(t+t')}{\cos\frac{1}{2}(t-t')}+\beta\frac{\sin\frac{1}{2}(t+t')}{\cos\frac{1}{2}(t-t')}$$

related to the points t and t' on the ellipse?

Ex. 4. The curve $\rho=\alpha t^2+2\beta t+\gamma$ is the trajectory of a point moving with uniform acceleration.

Ex. 5. What is the curve

$$\rho=\alpha\frac{1+t^2}{1-t^2}+2\beta\frac{t}{1-t^2}?$$

Investigate its properties.

Ex. 6. A helix is represented by

$$\rho=\alpha\cos t+\beta\sin t+\gamma t,$$

the vectors α, β and γ being mutually rectangular, and the tensors of α and β being equal. Determine all particulars.

Ex. 7. A conic is represented by the equation

$$\rho=\frac{\alpha t^2+2\beta t+\gamma}{at^2+2bt+c}.$$

Its centre is at the extremity of the vector

$$\kappa=\frac{\alpha c-2\beta b+\gamma a}{2(ac-b^2)}.$$

[The curve meets an arbitrary plane in two points. Find the pole of a chord, and in particular of the chord at infinity.]

Ex. 8. The equation $\quad V\rho\alpha V\beta\rho=(V\alpha\beta)^2$
represents a plane curve—a hyperbola of which α and β are the asymptotes.

Ex. 9. Write the equation of the conic of Ex. 7 in a vector form independent of the parameter.

ART. **49.** A vector function of two parameters, t and u,

$$\rho=\phi(t,\ u), \quad\text{.................................(I.)}$$

represents a surface. It may be regarded as generated by the family of curves $u=$ *constant*, t *variable*; or by the family $t=$ *const.*

In strict analogy with Art. 48, (VI.) we have

$$\left.\begin{aligned}d\rho=d\phi(t,\ u)&=\lim_{m=\infty,\ n=\infty} mn\left[\phi\left(t+\frac{1}{m}dt,\ u+\frac{1}{n}du\right)-\phi(t,\ u)\right]\\ &=\lim_{h=0,\ g=0}\frac{1}{hg}[\phi(t+hdt,\ u+gdu)-\phi(t,\ u)]\end{aligned}\right\},\ \text{(II.)}$$

where dt and du are any scalars. It is evident that this expression is linear with respect to dt and du, so that we may write

$$d\rho=d\phi(t,\ u)=\phi'.dt+\phi_{,}.du=\frac{\partial\phi}{\partial t}.dt+\frac{\partial\phi}{\partial u}.du. \quad\text{........(III.)}$$

The derived vectors $\phi'(t, u)$ and $\phi_{,}(t, u)$ are tangential respectively to the curves $u=\text{const.}$ and $t=\text{const.}$ at the point t, u; and more generally the vector $\phi' \mathrm{d}t+\phi_{,}\mathrm{d}u$ is tangential to the surface.

The equation of the tangent plane to the surface is

$$S(\rho-\phi)\phi'\phi_{,}=0, \text{ or } S(\rho-\phi)\nu=0, \text{ if } \nu \parallel V\phi'\phi_{,}, \text{ ....(IV.)}$$

and the vector ν is normal to the surface. The equation of the normal is

$$V(\rho-\phi)V\phi'\phi_{,}=0, \text{ or } V(\rho-\phi)\nu=0, \text{ or } \rho=\phi+x\nu. \text{ .....(V.)}$$

Ex. 1. If $\phi(t)$ is a function of a single parameter, the equation

$$\rho=\phi(t)+u\phi'(t)$$

represents a developable surface.

[This surface is generated by the tangent lines to the curve $\rho=\phi(t)$. The normal vector is $V(\phi'+u\phi'').\phi'$ or $V\phi''\phi'$, and is independent of u. The tangent plane is $S(\rho-\phi-u\phi')V\phi'\phi''=0$, or $S(\rho-\phi)\phi'\phi''=0$, and as this is independent of u, it touches the surface all along the generator determined by t. Conceive the tangent plane to roll over the surface and the successive generators to become attached to it, the surface will be unfolded or developed in the moving plane.]

Ex. 2. The equation

$$\rho=\phi(t)+u\alpha,$$

in which α is a constant vector, represents a cylinder standing on the curve $\rho=\phi(t)$ and having its generators parallel to α. The equation

$$\rho=u\phi(t)+\alpha$$

represents a cone standing on the same curve and having its vertex at the extremity of α.

Ex. 3. Find the locus of a line joining corresponding points on two homographically divided lines AB and CD.

[The surface is $\rho=\dfrac{\alpha+t\beta+s(l\gamma+tm\delta)}{1+t+s(l+tm)}$ if $\rho=\dfrac{\alpha+t\beta}{1+t}$, $\rho=\dfrac{l\gamma+tm\delta}{l+tm}$ are the homographically divided lines. This is a hyperboloid of one sheet.]

Ex. 4. Show that the variable line determines homographic divisions on the lines AC and BD.

Ex. 5. Find the scalar equation of the locus of Example 3, and show that it may be reduced to the form

$$XY=ZW,$$

where X, Y, Z and W are planes.

Ex. 6. Find the locus of a line similarly dividing two given lines AB and CD.

ART. **50.** The equation

$$\rho=\phi(t, u, v), \text{ ....(I.)}$$

in which t, u and v are variable parameters, may at pleasure be regarded as determining (I.) a singly infinite family of surfaces, for example, the surfaces found by assigning various but constant values to v; (II.) a doubly infinite family of curves, for example, t variable, u and v constant; (III.) any point in space, for we can in general find one or more sets of values of t, u, v corresponding to an arbitrary vector ρ. The scalars t, u, v are *curvilinear* coordinates of the extremity of the vector ρ.

Differential of a quaternion function.

ART. **51**. The differential of a quaternion function of a quaternion is defined by the equation

$$\mathrm{d}\,.\,F(q)=\lim_{n=\infty} n\left\{F\left(q+\frac{\mathrm{d}q}{n}\right)-Fq\right\}=f(\mathrm{d}q), \quad \text{(I.)}$$

or

$$\mathrm{d}\,.\,F(q)=\lim_{h=0}\frac{1}{h}\{F(q+h\mathrm{d}q)-Fq\}=f(\mathrm{d}q),$$

a definition in complete agreement with the results of Art. 48.

The function $f(\mathrm{d}q)$ *is a linear and distributive function of the differential* $\mathrm{d}q$, while it also in general involves the quaternion q in its constitution. To prove this proposition, observe that if r and s are any two quaternions,

$$\begin{aligned} f(r+s)&=\lim_{n=\infty} n\left\{F\left(q+\frac{r+s}{n}\right)-Fq\right\}\\ &=\lim_{n=\infty} n\left\{F\left(q+\frac{r+s}{n}\right)-F\left(q+\frac{s}{n}\right)+F\left(q+\frac{s}{n}\right)-Fq\right\}\\ &=\lim_{n=\infty} n\left\{F\left(q+\frac{r+s}{n}\right)-F\left(q+\frac{s}{n}\right)\right\}+\lim_{n=\infty} n\left\{F\left(q+\frac{s}{n}\right)-Fq\right\}\\ &=\lim_{n=\infty} n\left\{F\left(q+\frac{r}{n}\right)-F(q)\right\}+\lim_{n=\infty} n\left\{F\left(q+\frac{s}{n}\right)-Fq\right\}, \end{aligned}$$

or simply $\quad f(r+s)=f(r)+f(s). \quad$ (II.)

As a corollary, $\quad f(xr)=xf(r), \quad$ (III.)

if x is any scalar.

As an example,

$$\begin{aligned} \mathrm{d}\,.\,q^2&=\lim_{n=\infty} n\left\{\left(q+\frac{\mathrm{d}q}{n}\right)^2-q^2\right\}=\lim_{n=\infty} n\left\{q^2+q\,.\,\frac{\mathrm{d}q}{n}+\frac{\mathrm{d}q}{n}q+\frac{(\mathrm{d}q)^2}{n^2}-q^2\right\}\\ &=\lim_{n=\infty}\left\{q\,.\,\mathrm{d}q+\mathrm{d}q\,.\,q+\frac{\mathrm{d}q^2}{n}\right\}, \end{aligned}$$

and thus $\quad \mathrm{d}\,.\,q^2=q\,.\,\mathrm{d}q+\mathrm{d}q\,.\,q. \quad$ (IV.)

There is a notable difference between the differential of a function of a single scalar and a function of a quaternion, which is clearly illustrated by this example. In general, from a differential of a function of a single scalar $\mathrm{d} \,.\, F(x)$, we can form a differential coefficient $\dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$, which is absolutely independent of $\mathrm{d}x$. Thus, $\dfrac{\mathrm{d} \,.\, x^2}{\mathrm{d}x} = 2x$, but $\dfrac{\mathrm{d} \,.\, q^2}{\mathrm{d}q} = q + \mathrm{d}q \,.\, q \,.\, \mathrm{d}q^{-1}$ is not independent of $\mathrm{d}q$. And this, which is a consequence of the non-commutative law of multiplication, is really quite in keeping with the ordinary theory, for if $F(x, y)$ is a function of two independent scalars x and y, we cannot form a complete differential coefficient from $\mathrm{d} \,.\, F(xy) = \dfrac{\partial F}{\partial x}\mathrm{d}x + \dfrac{\partial F}{\partial y}\mathrm{d}y$, where $\mathrm{d}x$ and $\mathrm{d}y$ are arbitrary, though we can of course form the partial differential coefficients $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$. We must remember that a quaternion is a function of four numbers, and that a differential $\mathrm{d}q$ is susceptible of a quadruply infinite system of values.

As a second example,

$$\mathrm{d} \,.\, q^{-1} = -q^{-1} \,.\, \mathrm{d}q \,.\, q^{-1}, \qquad \text{(v.)}$$

for

$$\mathrm{d} \,.\, q^{-1} = \lim_{n=\infty} n\left\{\left(q + \frac{1}{n}\mathrm{d}q\right)^{-1} - q^{-1}\right\}$$

$$= \lim_{n=\infty} n \,.\, \left(q + \frac{1}{n}\mathrm{d}q\right)^{-1}\left\{1 - \left(q + \frac{1}{n}\mathrm{d}q\right)q^{-1}\right\}$$

$$= \lim_{n=\infty} \left(q + \frac{1}{n}\mathrm{d}q\right)^{-1} \,.\, \mathrm{d}q \,.\, q^{-1}.$$

Ex. 1. Prove that

$$\mathrm{d} \,.\, \mathrm{S}q = \mathrm{S}\mathrm{d}q, \quad \mathrm{d}\mathrm{V}q = \mathrm{V}\mathrm{d}q, \quad \mathrm{d}\mathrm{K}q = \mathrm{K}\mathrm{d}q.$$

[Note that these symbols are distributive, or that

$$\mathrm{S}(q + n^{-1}\mathrm{d}q) = \mathrm{S}q + n^{-1}\mathrm{S}\mathrm{d}q.]$$

Ex. 2. If ν is a vector function of a variable vector ρ, and if $\mathrm{d}\nu = \phi d\rho$ show that $\phi \mathrm{d}\rho$ is a linear and distributive vector function of $\mathrm{d}\phi$, so that for any pair of vectors $\phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta)$.

[This is a particular case of (II.). Fuller details will be found in the following chapter.]

ART. **52.** The differential of a function $F(q, r, s, \ldots)$ of any number of quaternions is the sum of the differentials with respect to each separately, or

$$\mathrm{d} \,.\, F(q, r, s, \ldots) = \mathrm{d}_q \,.\, F(q, r, s, \ldots) + \mathrm{d}_r \,.\, F(q, r, s, \ldots) + \text{etc.}, \ldots \text{(I.)}$$

where $\mathrm{d}_q \,.\, F(q, r, s, \ldots)$ denotes the differential of the function on the supposition that q alone is variable. We may write

$$\mathrm{d} \,.\, F(q, r, s, \ldots)$$
$$= \lim_{n=\infty} n\left\{F\left(q+\frac{1}{n}\mathrm{d}q,\ r+\frac{1}{n}\mathrm{d}r,\ s+\frac{1}{n}\mathrm{d}s, \ldots\right)-F\left(q, r, s, \ldots\right)\right\}, \quad \text{(II.)}$$

and this, by the process of the last article, leads at once to (I.).

Thus,

$$\mathrm{d} \,.\, qr = \mathrm{d}q \,.\, r + q \,.\, \mathrm{d}r,$$
$$\mathrm{d} \,.\, qq^{-1} = 0 = \mathrm{d}q \,.\, q^{-1} + q \,.\, \mathrm{d} \,.\, q^{-1}, \quad \mathrm{d} \,.\, q^{-1} = -q^{-1} \,.\, \mathrm{d}q \,.\, q^{-1}.$$

Generally in any product of variable quaternions, the rule is to differentiate each quaternion in the position it occupies.

Ex. 1. Differentiate $r = aqbqc$, where q is variable.

Ex. 2. Differentiate $(qr)^2$ and q^2r^2, where q and r are both variable.

ART. **53.** The differentials of the functions Sq, Vq, Kq, Uq, Tq, UVq, etc., of a quaternion are naturally of importance. We have already stated that

$$\mathrm{dS}q = \mathrm{Sd}q, \quad \mathrm{dV}q = \mathrm{Vd}q, \quad \mathrm{dK}q = \mathrm{Kd}q, \quad \text{(I.)}$$

and these results are immediate consequences of the distributive character of the symbols, S, V, K.

Since (Art. 17, p. 12)

$\mathrm{T}q^2 = q\mathrm{K}q$, we have $2\mathrm{T}q \,.\, \mathrm{dT}q = \mathrm{d}q \,.\, \mathrm{K}q + q \,.\, \mathrm{Kd}q = 2\mathrm{Sd}q\mathrm{K}q$ (compare Ex. 6, Art. 20, p. 15), and since $\mathrm{K}q = \mathrm{T}q(\mathrm{U}q)^{-1}$, the differential of T$q$ is

$$\mathrm{dT}q = \mathrm{S}\frac{\mathrm{d}q}{\mathrm{U}q}, \text{ or } \frac{\mathrm{dT}q}{\mathrm{T}q} = \mathrm{S}\frac{\mathrm{d}q}{q}. \quad \text{(II.)}$$

Further, since

$$q = \mathrm{T}q \,.\, \mathrm{U}q, \text{ and } \mathrm{d}q = \mathrm{dT}q \,.\, \mathrm{U}q + \mathrm{T}q \,.\, \mathrm{dU}q,$$

we have on division by q,

$$\frac{\mathrm{d}q}{q} = \frac{\mathrm{dT}q}{\mathrm{T}q} + \frac{\mathrm{dU}q}{\mathrm{U}q}, \quad \text{(III.)}$$

and therefore by (II.), $\dfrac{\mathrm{dU}q}{\mathrm{U}q} = \mathrm{V} \,.\, \dfrac{\mathrm{d}q}{q}$. (IV.)

In particular for vectors,

$$\mathrm{dT}\rho^2 = -\mathrm{d} \,.\, \rho^2 = -2\mathrm{S}\rho\mathrm{d}\rho = 2\mathrm{T}\rho^2\mathrm{S}\rho^{-1}\mathrm{d}\rho$$

and $\mathrm{d}\rho = \mathrm{T}\rho \,.\, \mathrm{dU}\rho + \mathrm{U}\rho \,.\, \mathrm{dT}\rho$, and therefore,

$$\frac{\mathrm{dT}\rho}{\mathrm{T}\rho} = \mathrm{S}\frac{\mathrm{d}\rho}{\rho}, \quad \frac{\mathrm{dU}\rho}{\mathrm{U}\rho} = \mathrm{V}\frac{\mathrm{d}\rho}{\rho}. \quad \text{(V.)}$$

The relations $$\mathrm{S}\frac{\mathrm{dU}q}{\mathrm{U}q}=0, \quad \mathrm{S}\frac{\mathrm{dU}\rho}{\mathrm{U}\rho}=0 \quad \text{.....................(VI.)}$$
are worthy of notice.

Ex. 1. Resolve $\mathrm{d}\rho$ into components along and perpendicular to ρ.

Ex. 2. If $\rho=r\alpha^{\frac{2u}{\pi}}\beta$, where $\mathrm{T}\alpha=\mathrm{T}\beta=1$, $\mathrm{S}\alpha\beta=0$, and where the scalars r and u alone vary, show that
$$\mathrm{V}\frac{\mathrm{d}\rho}{\rho}=\alpha\mathrm{d}u, \quad \mathrm{S}\frac{\mathrm{d}\rho}{\rho}=\frac{\mathrm{d}r}{r}.$$

(a) Prove generally that $\mathrm{TV}\,.\,\mathrm{d}\rho\rho^{-1}$ is the differential of the angle swept out by the varying vector $\mathrm{OP}=\rho$.

Ex. 3. If ρ and ρ' are inverse points, the origin being the centre of inversion, and if $\mathrm{d}\rho$ and $\mathrm{d}'\rho$ are any two differentials of ρ, and $\mathrm{d}\rho'$ and $\mathrm{d}'\rho'$ the corresponding differentials of ρ', prove that
$$\frac{\mathrm{d}\rho'}{\mathrm{d}'\rho'}=\rho^{-1}\,.\,\frac{\mathrm{d}\rho}{\mathrm{d}'\rho}\,.\,\rho,$$
and interpret the meaning of this relation.

Ex. 4. Compare an element of vector area with the corresponding element into which it is changed by inversion.

[The elements are $\mathrm{Vd}\rho\mathrm{d}'\rho$ and $R^4\mathrm{T}\rho^{-4}\,.\,\rho^{-1}\mathrm{Vd}\rho\mathrm{d}'\rho\,.\,\rho$.]

Ex. 5. Prove that

$$(a)\quad \mathrm{dUV}q=\mathrm{V}\frac{\mathrm{Vd}q}{\mathrm{V}q}\,.\,\mathrm{UV}q.$$

$$(b)\quad \mathrm{dVU}q=\mathrm{V}\left(\mathrm{V}\frac{\mathrm{d}q}{q}\,.\,\mathrm{U}q\right).$$

$$(c)\quad \mathrm{dSU}q=\mathrm{S}\left(\mathrm{V}\frac{\mathrm{d}q}{q}\,.\,\mathrm{U}q\right).$$

$$(d)\quad \mathrm{d}\angle q=\mathrm{S}\left(\frac{\mathrm{d}q}{\mathrm{UV}q\,.\,q}\right).$$

Ex. 6. The vector α being constant, prove that
$$\mathrm{d}\,.\,q\alpha q^{-1}=2\mathrm{V}\,.\,\mathrm{Vd}qq^{-1}\,.\,q\alpha q^{-1}=2q(\mathrm{V}\,.\,\mathrm{V}q^{-1}\mathrm{d}q\,.\,\alpha)q^{-1}.$$

Ex. 7. Prove that
$$\mathrm{d}\alpha^x=\mathrm{d}x\left(\log\mathrm{T}\alpha+\frac{\pi}{2}\mathrm{U}\alpha\right)\alpha^x,$$
where α is a constant vector and x a variable scalar; and that
$$\mathrm{d}\alpha^x=x\mathrm{S}\frac{\mathrm{d}\alpha}{\alpha}\,.\,\alpha^x+\mathrm{V}\frac{\mathrm{d}\alpha}{\alpha}\,.\,\mathrm{V}\alpha^x,$$
where x is constant and α variable.

ART. **54.** If P is any scalar function of a variable vector ρ, a differential of P is connected with the corresponding differential of ρ by a relation of the form
$$\mathrm{d}P=-\mathrm{S}\nu\mathrm{d}\rho, \quad \text{.................................(I.)}$$
the vector ν being a function of ρ but independent of $\mathrm{d}\rho$.

The rate of variation of P along any direction α $(\mathrm{T}\alpha=1)$, may be written in the form

$$\mathrm{d}_\alpha P = -\mathrm{S}\nu\alpha, \qquad \text{(II.)}$$

it being understood that the suffix α attached to d signifies that the corresponding differential of ρ is

$$\mathrm{d}\rho=\alpha. \qquad \text{(III.)}$$

This rate of variation as expressed by (II.) is the projection of the vector ν along the vector α, and consequently the rate of variation of P is maximum along the vector ν, being then equal to $\mathrm{T}\nu$, while it is zero along any direction normal to ν.

Having given the variations of P along three non-coplanar directions, or what is equivalent, having given the differentials $\mathrm{d}P$, $\mathrm{d}'P$ and $\mathrm{d}''P$ of P corresponding to three non-coplanar differentials $\mathrm{d}\rho$, $\mathrm{d}'\rho$ and $\mathrm{d}''\rho$ of ρ, we can determine the vector ν. We have in fact

$$\mathrm{d}P=-\mathrm{S}\nu\mathrm{d}\rho,\quad \mathrm{d}'P=-\mathrm{S}\nu\mathrm{d}'\rho,\quad \mathrm{d}''P=-\mathrm{S}\nu\mathrm{d}''\rho, \qquad \text{(IV.)}$$

and by the fundamental formula of Art. 26, p. 24, we find

$$\nu=-\frac{\mathrm{V}\mathrm{d}'\rho\mathrm{d}''\rho\,.\,\mathrm{d}P+\mathrm{V}\mathrm{d}''\rho\mathrm{d}\rho\,.\,\mathrm{d}'P+\mathrm{V}\mathrm{d}\rho\mathrm{d}'\rho\,.\,\mathrm{d}''P}{\mathrm{S}\mathrm{d}\rho\mathrm{d}'\rho\mathrm{d}''\rho}. \qquad \text{(V.)}$$

Thus it appears that the vector ν is derived from P by means of the differentiating operator

$$\nabla=-\frac{\mathrm{V}\mathrm{d}'\rho\mathrm{d}''\rho\,.\,\mathrm{d}+\mathrm{V}\mathrm{d}''\rho\mathrm{d}\rho\,.\,\mathrm{d}'+\mathrm{V}\mathrm{d}\rho\mathrm{d}'\rho\,.\,\mathrm{d}''}{\mathrm{S}\mathrm{d}\rho\mathrm{d}'\rho\mathrm{d}''\rho}, \qquad \text{(VI.)}$$

in which $\mathrm{d}\rho$, $\mathrm{d}'\rho$ and $\mathrm{d}''\rho$ are any three non-coplanar differentials of ρ, and in which d, d′ and d″ are the corresponding symbols of differentiation.

Ex. 1. Prove that

$$\nabla\mathrm{S}\alpha\rho=-\alpha,$$
$$\nabla\rho^2=-2\rho,$$
$$\nabla\mathrm{T}\rho=+\mathrm{U}\rho,$$
$$\nabla\mathrm{T}\mathrm{V}\alpha\rho=+\mathrm{U}\mathrm{V}\alpha\rho\,.\,\alpha,$$
$$\nabla\mathrm{T}(\rho-\alpha)^{-1}=-\mathrm{U}(\rho-\alpha)\,.\,\mathrm{T}(\rho-\alpha)^{-2}.$$

[These follow from the relation $\mathrm{d}P=-\mathrm{S}\mathrm{d}\rho\nabla P$.]

Ex. 2. Show that

$$\mathrm{S}\alpha\nabla\,.\,\mathrm{T}\rho^{-1}=-\mathrm{S}\alpha\rho\,.\,\mathrm{T}\rho^{-3},$$
$$\mathrm{S}\beta\nabla\mathrm{S}\alpha\nabla\,.\,\mathrm{T}\rho^{-1}=3\mathrm{S}\alpha\rho\mathrm{S}\beta\rho\,.\,\mathrm{T}\rho^{-5}+\mathrm{S}\alpha\beta\,.\,\mathrm{T}\rho^{-3},$$
$$\mathrm{S}\gamma\nabla\mathrm{S}\beta\nabla\mathrm{S}\alpha\nabla\,.\,\mathrm{T}\rho^{-1}=-3\,.\,5\mathrm{S}\alpha\rho\mathrm{S}\beta\rho\mathrm{S}\gamma\rho\,.\,\mathrm{T}\rho^{-7}-3\Sigma\mathrm{S}\beta\gamma\mathrm{S}\alpha\rho\,.\,\mathrm{T}\rho^{-5}.$$

ART. **55**. The form of the expression found in the last article for ∇P suggests a new view of the subject which is applicable in the general case when P is a vector or even a quaternion function of ρ. Suppose a parallelepiped constructed having its edges equal to any three vectors $\mathrm{d}\rho$, $\mathrm{d}'\rho$ and $\mathrm{d}''\rho$, and having its centre at the extremity of ρ. If we suppose the vectors arranged in positive order of rotation (compare Art. 24), $Vd'\rho d''\rho$ is the outwardly directed vector area of the face having its centre at the extremity of $\rho+\frac{1}{2}d\rho$; and $-Vd'\rho d''\rho$ is likewise the outwardly directed area of the face, centre $\rho-\frac{1}{2}d\rho$. Also $-Sd\rho d'\rho d''\rho$ is the volume of the parallelepiped.

Let $F(\rho)$ be any function of ρ, scalar, vector or quaternion, then the sum of the products of the outwardly directed vector faces into the value of $F(\rho)$ at their middle points is

$$Vd'\rho d''\rho . F(\rho+\tfrac{1}{2}d\rho)+Vd''\rho d\rho . F(\rho+\tfrac{1}{2}d'\rho)+Vd\rho d'\rho . F(\rho+\tfrac{1}{2}d''\rho)$$
$$-Vd'\rho d''\rho . F(\rho-\tfrac{1}{2}d\rho)-Vd''\rho d\rho . F(\rho-\tfrac{1}{2}d'\rho)-Vd\rho d'\rho . F(\rho-\tfrac{1}{2}d''\rho), \quad \text{(I.)}$$

and the quotient of this sum by the volume of the parallelepiped is

$$\frac{\Sigma Vd'\rho d''\rho . \{(F(\rho+\frac{1}{2}d\rho)-F(\rho-\frac{1}{2}d\rho)\}}{-Sd\rho d'\rho d''\rho}. \quad \text{(II.)}$$

Each edge being diminished in the ratio $\frac{1}{n}$, the quotient becomes

$$\frac{n^{-2}\Sigma Vd'\rho d''\rho\left\{F\left(\rho+\frac{1}{2n}d\rho\right)-F\left(\rho-\frac{1}{2n}d\rho\right)\right\}}{-n^{-3}Sd\rho d'\rho d''\rho}. \quad \text{(III.)}$$

So that when n increases without limit, or when the parallelepiped whose edges are $\frac{1}{n}d\rho$, $\frac{1}{n}d'\rho$, $\frac{1}{n}d''\rho$ decreases without limit, the limiting value of the quotient (III.) is (compare Art. 51 (I.))

$$-\lim_{n=\infty}\frac{\Sigma Vd'\rho d''\rho . n\left\{F\left(\rho+\frac{1}{2n}d\rho\right)-F\left(\rho-\frac{1}{2n}d\rho\right)\right\}}{Sd\rho d'\rho d''\rho}$$
$$=-\frac{\Sigma Vd'\rho d''\rho . dF\rho}{Sd\rho d'\rho d''\rho}=\nabla . F\rho. \quad \text{(IV.)}$$

Thus $\nabla . F(\rho)$ is the limit of the ratio which the sum of the products of the outwardly directed faces of a parallelepiped into the mean values of $F(\rho)$ over the faces bears to the volume of the parallelepiped. And the vectors $d\rho$, $d'\rho$, $d''\rho$ being arbitrary, the result is independent of the shape of the parallelepiped.

Take the case in which $F(\rho)$ is a vector function (σ) of ρ, and consider separately the scalar and the vector parts of $\nabla . \sigma$. The scalar part is the limit of the ratio which the sum of the scalar products of σ into the outwardly directed elements of the surface—or which the sum of the inwardly directed normal components of σ into the corresponding area*—or which the surface

* Remember that $S\alpha\beta$ is *minus* the length of one vector into the projection of the other upon it.

integral of the inward normal component of σ—bears to the volume. Thus if σ represents the flux of a fluid, $S\nabla\sigma$ is the rate per unit volume at which the amount of the fluid is increasing at the point in unit time. In other words $S\nabla\sigma$ is the rate of increase of the density at the point. If σ is the *velocity* of a fluid and c the density, $c\sigma$ is the flux, or the mass of the fluid that crosses unit area normal to σ in unit time, and $S\nabla.(c\sigma)$ is the rate of increase of density at the point, or $\frac{\partial c}{\partial t}$. Thus

$$\frac{\partial c}{\partial t} = S\nabla(c\sigma). \qquad \text{(v.)}$$

For an incompressible fluid, c is constant and $S\nabla\sigma$ is zero.

In like manner, $V.\nabla\sigma$ is the limit of the ratio borne to the volume by the integral over the surface of the vector product $V.U\nu.\sigma.dA$, where $U\nu$ is the outwardly directed unit vector along the normal and dA the scalar element of area, or where $U\nu dA$ is the outwardly directed vector element of area.

Since it has appeared that these results are independent of the shape of the parallelepiped, it follows that they are true for any closed surface formed of a single sheet, and we have

$$\lim \frac{\int d\nu F(\rho)}{v} = \nabla . F(\rho), \qquad \text{(vi.)}$$

where $d\nu$ is an outwardly directed element of vector area of the surface, and where v is the volume, the limit being arrived at when the surface becomes indefinitely small.

ART. **56.** Towards further elucidation of the operator ∇, consider the analogous integral taken round the vector sides of a parallelogram, having its centre at the extremity of the vector ρ.

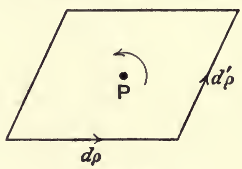

FIG. 24.

Circuiting in the positive direction and forming the product of the vector sides into the corresponding values of $F(\rho)$ at their middle points, the sum is

$$d\rho.F(\rho-\tfrac{1}{2}d'\rho)+d'\rho.F(\rho+\tfrac{1}{2}d\rho)-d\rho F(\rho+\tfrac{1}{2}d'\rho)-d'\rho F(\rho-\tfrac{1}{2}d\rho).$$

Collecting terms and dividing by the area of the parallelogram, the result is

$$\frac{d'\rho.\{F(\rho+\frac{1}{2}d\rho)-F(\rho-\frac{1}{2}d\rho)\}-d\rho\{F(\rho+\frac{1}{2}d'\rho)-F(\rho-\frac{1}{2}d'\rho)\}}{TVd\rho d'\rho}.$$

Now let the parallelogram be indefinitely diminished by replacing $d\rho$ and $d'\rho$ by $\frac{1}{n}d\rho$ and $\frac{1}{n}d'\rho$, and we have in the limit,

$$\lim_{n=\infty} \frac{d'\rho\left\{F\left(\rho+\frac{d\rho}{2n}\right)-F\left(\rho-\frac{d\rho}{2n}\right)\right\}-d\rho\left\{F\left(\rho+\frac{d'\rho}{2n}\right)-F\left(\rho-\frac{d'\rho}{2n}\right)\right\}}{n^{-1}TVd\rho d'\rho}$$

$$=\frac{d'\rho\,.\,dF\rho-d\rho\,.\,d'F\rho}{TVd\rho d'\rho}. \quad \text{(I.)}$$

But this is equal to

$$\frac{\{V\,.\,Vd\rho d'\rho(Vd'\rho d''\rho\,.\,d\,.+Vd''\rho d\rho\,.\,d'\,.+Vd\rho d'\rho\,.\,d''\,.)\}F\rho}{-Sd\rho d'\rho d''\rho TVd\rho d'\rho}$$

because $V(Vd\rho d'\rho\,.\,Vd'\rho d''\rho)=-d'\rho Sd\rho d'\rho d''\rho$, etc., so that the integral is

$$\frac{V(Vd\rho d'\rho\,.\,\nabla)}{TVd\rho d'\rho}\,.\,F\rho=V(U\nu\,.\,\nabla)\,.\,F\rho \quad \text{(II.)}$$

if $U\nu=UVd\rho d'\rho$ is the normal to the area about which the direction of circuiting is positive.

As in the last article, we have for any plane closed curve without loops,

$$\lim\frac{\int d\rho F(\rho)}{A}=V(U\nu\,.\,\nabla)\,.\,F(\rho), \quad \text{(III.)}$$

$d\rho$ being now a vector element of arc of the curve and A being its scalar area.

In particular for a vector function (σ) of ρ, we have separately

$$\lim\frac{\int Sd\rho\sigma}{A}=S(VU\nu\nabla\,.\,\sigma),\quad \lim\frac{\int Vd\rho\sigma}{A}=V(VU\nu\nabla\,.\,\sigma)\text{....(IV.)}$$

It is obvious on using the expanded form of ∇ that we may write

$$S(VU\nu\nabla\,.\,\sigma)=S(U\nu V\nabla\sigma)=SU\nu\nabla\sigma, \quad \text{(V.)}$$

or that we may in this relation at least treat ∇ as a vector in combination with other vectors, it being understood that ∇ operates on σ but not on $U\nu$.

This result leads us back to an interpretation of $V\nabla\sigma$ analogous to the interpretation of ∇P in Art. 54. We have

$$SU\nu V\nabla\sigma=\lim\frac{\int Sd\rho\sigma}{A}, \quad \text{(VI.)}$$

or the limit of the ratio which the integrated component of σ along the arc of a plane curve ($-\int Sd\rho\sigma$) bears to the area of that curve, is equal to the component ($-SU\nu V\nabla\sigma$) of the vector $V\nabla\sigma$

along the positive normal to the plane. This is a maximum and equal to $TV\nabla\sigma$ when the plane is at right angles to $UV\nabla\sigma$; it vanishes when the plane is parallel to that direction.

If $Sd\rho\sigma$ is the differential of P (some scalar function of ρ), the integral $\int Sd\rho\sigma$ depends merely on the limits between which the integral is taken (leaving aside cases in which singularities occur), and is in fact $P(\rho_2)-P(\rho_1)$ if the integration extends from ρ_1 to ρ_2. For any small closed circuit therefore the integral vanishes, the initial and final points of the path of integration being coincident, and therefore

$$V\nabla\sigma=0, \text{ if } S\sigma d\rho=dP. \quad \text{(VII.)}$$

Conversely, if $V\nabla\sigma=0$, we must have $S\sigma d\rho$ the differential of a scalar P; for in this case the integral round any small closed circuit vanishes, or what is equivalent, the integral from ρ_1 to ρ_2 is equal and opposite to the integral back by another path from ρ_2 to ρ_1, or again, the integral from ρ_1 to ρ_2 is independent of the path. These results will be extended to the general case of curves which are not small. At present we remark that

$$V\nabla\nabla P=0, \text{ or } V\nabla^2P=0, \text{ or } \nabla^2P=\textit{scalar}, \quad \text{(VIII.)}$$

if P is a scalar function of ρ, is involved in equation (VII.).

ART. **57**. It is useful to express the operator ∇ in various forms. If, for example, as in Art. 50, we suppose the vector ρ to be expressed in terms of three parameters u, v and w, and if we write

$$d\rho=\frac{\partial\rho}{\partial u}\cdot du=\rho_1 du, \quad d'\rho=\frac{\partial\rho}{\partial v}dv=\rho_2 dv, \quad d''\rho=\frac{\partial\rho}{\partial w}dw=\rho_3 dw, \quad \text{(I.)}$$

the symbols of differentiation d, d′ and d″ refer respectively to u, v and w, so that symbolically

$$d=\frac{\partial}{\partial u}\cdot du, \quad d'=\frac{\partial}{\partial v}\cdot dv, \quad d''=\frac{\partial}{\partial w}\cdot dw. \quad \text{(II.)}$$

On this understanding, equation (VI.), Art. 54, becomes

$$\nabla=-\frac{V\rho_2\rho_3\cdot\frac{\partial}{\partial u}+V\rho_3\rho_1\cdot\frac{\partial}{\partial v}+V\rho_1\rho_2\cdot\frac{\partial}{\partial w}}{S\rho_1\rho_2\rho_3}. \quad \text{(III.)}$$

If the parameters are so selected that the derived vectors ρ_1, ρ_2 and ρ_3 are always mutually perpendicular, the symbols V and S in (III.) become superfluous, and the expression for ∇ reduces to the simple form,

$$\nabla=-\rho_1{}^{-1}\frac{\partial}{\partial u}-\rho_2{}^{-1}\frac{\partial}{\partial v}-\rho_3{}^{-1}\frac{\partial}{\partial w}. \quad \text{(IV.)}$$

If the vector ρ is expressed in terms of the Cartesian coordinates x, y and z, so that $\rho = ix + jy + kz$, we have $\rho_1 = i$, $\rho_2 = j$, $\rho_3 = k$, and

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}. \quad \text{.......................(v.)}$$

This last form may be regarded as the canonical form of the operator. We have, for example, when q is the operand,

$$\nabla . q = i\frac{\partial q}{\partial x} + j\frac{\partial q}{\partial y} + k\frac{\partial q}{\partial z} = \Sigma i\frac{\partial \mathrm{S}q}{\partial x} + \Sigma i\frac{\partial \mathrm{V}q}{\partial x} = \nabla \mathrm{S}q + \mathrm{S}\nabla q + \mathrm{V}\nabla \mathrm{V}q;$$

and we shall write

$$q . \nabla = \frac{\partial q}{\partial x} . i + \frac{\partial q}{\partial y} . j + \frac{\partial q}{\partial z} . k = \Sigma i\frac{\partial \mathrm{S}q}{\partial x} + \Sigma\frac{\partial \mathrm{V}q}{\partial x} . i = \nabla \mathrm{S}q + \mathrm{S}\nabla q - \mathrm{V}\nabla \mathrm{V}q;$$

so that in combination with its operand ∇ acts as a vector in combination with a quaternion.

Again if a is a constant quaternion, we have symbolically, an operand being understood,

$$\nabla . a = ia\frac{\partial}{\partial x} + ja\frac{\partial}{\partial y} + ka\frac{\partial}{\partial z} = \nabla . \mathrm{S}a + \mathrm{S}\nabla a + \mathrm{V}\nabla \mathrm{V}a,$$

$$a . \nabla = ai\frac{\partial}{\partial x} + aj\frac{\partial}{\partial y} + ak\frac{\partial}{\partial z} = \nabla . \mathrm{S}a + \mathrm{S}\nabla a - \mathrm{V}\nabla \mathrm{V}a;$$

and in combination with a quaternion, not the operand, ∇ still plays the rôle of a vector.

In combination with itself

$$\begin{aligned}\nabla . \nabla . q &= \nabla . \left(i\frac{\partial q}{\partial x} + j\frac{\partial q}{\partial y} + k\frac{\partial q}{\partial z}\right) \\ &= i^2\frac{\partial^2 q}{\partial x^2} + j^2\frac{\partial^2 q}{\partial y^2} + k^2\frac{\partial^2 q}{\partial z^2} + jk\frac{\partial^2 q}{\partial y \partial z} + kj\frac{\partial^2 q}{\partial z \partial y} \\ &\qquad + ki\frac{\partial^2 q}{\partial z \partial x} + ik\frac{\partial^2 q}{\partial x \partial z} + ij\frac{\partial^2 q}{\partial x \partial y} + ji\frac{\partial^2 q}{\partial y \partial x} \\ &= -\frac{\partial^2 q}{\partial x^2} - \frac{\partial^2 q}{\partial y^2} - \frac{\partial^2 q}{\partial z^2} = +\nabla^2 . q,\end{aligned}$$

and generally in all combinations ∇ may be treated as a symbolic vector. Of course some little care is necessary when ∇ is expressed in the general form, but it is precisely of the same kind as the care required to distinguish between

$$\left(x^2\frac{\partial}{\partial x}\right)^2 = x^4\frac{\partial^2}{\partial x^2} + 2x^3\frac{\partial}{\partial x} \quad \text{and} \quad x^4\left(\frac{\partial}{\partial x}\right)^2.$$

Ex. 1. Show that if $q=W+iX+jY+kZ$,

$$\nabla q=-\frac{\partial X}{\partial x}-\frac{\partial Y}{\partial y}-\frac{\partial Z}{\partial z}+i\left(\frac{\partial W}{\partial x}+\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)$$
$$+j\left(\frac{\partial W}{\partial y}+\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\right)+k\left(\frac{\partial W}{\partial z}+\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right).$$

Ex. 2. Verify that

$$\nabla\,.\,\nabla\sigma=\nabla^2\,.\,\sigma=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\sigma,\quad\text{where}\quad \sigma=iX+jY+kZ.$$

Ex. 3. Prove that $\nabla\rho=-3$; $\nabla\mathrm{V}\lambda\rho=2\lambda$; $\nabla\mathrm{U}\rho=-2\mathrm{T}\rho^{-1}$; $\nabla\rho^{-1}=\mathrm{T}\rho^{-2}$; $\nabla^2\,.\,\mathrm{T}(\rho-\lambda)^{-1}=0$ if ρ is not equal to λ; $\nabla^2\mathrm{TV}\lambda\rho=-\mathrm{T}(\mathrm{V}\lambda^{-1}\rho)^{-1}$;

$$\nabla^2\log\mathrm{TV}\lambda\rho=0;\quad \nabla^2f\mathrm{T}\rho=-f''\mathrm{T}\rho-2\mathrm{T}\rho^{-1}f'\mathrm{T}\rho.$$

[For example, $\nabla\mathrm{V}\lambda\rho=-\dfrac{\Sigma\mathrm{V}\beta\gamma\mathrm{V}\lambda\alpha}{\mathrm{S}\alpha\beta\gamma}=-\dfrac{\mathrm{V}\beta\lambda\mathrm{V}\lambda\alpha-\mathrm{V}\alpha\lambda\mathrm{V}\lambda\beta}{\mathrm{S}\alpha\beta\lambda}$.]

Ex. 4. Prove that $\mathrm{V}\lambda\nabla\,.\,\rho=-2\lambda$; $\mathrm{V}\nabla\mathrm{V}\lambda\nabla\,.\,P=-\lambda\nabla^2P+\nabla\mathrm{S}\lambda\nabla\,.\,P$.

Ex. 5. Show that

$$(\alpha\nabla+\nabla\alpha)q=2\mathrm{S}\alpha\nabla\,.\,q,\quad (\alpha\nabla-\nabla\alpha)q=2\mathrm{V}\alpha\nabla\,.\,q.$$

[Here $(\alpha\nabla+\nabla\alpha)\,.\,q=-\Sigma\dfrac{\alpha\mathrm{Vd}'\rho\mathrm{d}''\rho+\mathrm{Vd}'\rho\mathrm{d}''\rho\,.\,\alpha}{\mathrm{Sd}\rho\mathrm{d}'\rho\mathrm{d}''\rho}\,.\,\mathrm{d}q=-2\Sigma\dfrac{\mathrm{S}\alpha\mathrm{d}'\rho\mathrm{d}''\rho}{\mathrm{Sd}\rho\mathrm{d}'\rho\mathrm{d}''\rho}\,.\,\mathrm{d}q.$]

Ex. 6. If P and Q are scalar functions of ρ, show that

$$\nabla\,.\,PQ=\nabla P\,.\,Q+\nabla Q\,.\,P.$$

Ex. 7. If p and q are quaternion functions of ρ, show that

$$\nabla\,.\,pq=\nabla p\,.\,q_0+\nabla\,.\,p_0\,.\,q,$$

where the suffix is intended to denote that the affected symbols are not to be operated on by ∇.

Ex. 8. Interpret the expressions

$$\mathrm{V}\nabla\nabla'\,.\,PQ',\quad \mathrm{S}\nabla\nabla'\nabla''\,.\,PQ'R'',$$

where the accents indicate that a marked symbol is to be operated on by the correspondingly marked ∇.

[If P and Q are scalars, the first expression is $\mathrm{V}(\nabla P)(\nabla Q)$, or

$$\Sigma i\left(\frac{\partial P}{\partial y}\frac{\partial Q}{\partial z}-\frac{\partial P}{\partial z}\frac{\partial Q}{\partial y}\right).$$

This last expression is also true when P and Q are quaternions.]

Ex. 9. Find an expanded form for $\nabla^2\,.\,PQ$.

Ex. 10. Find the expression for ∇ in terms of the usual r, θ and ϕ coordinates. [Use the relation (IV.).]

Ex. 11. Show that $q\,.\,\nabla=-\mathrm{K}\,.\,\nabla\mathrm{K}q$ where ∇ operates on q *in situ*.

[It is sometimes convenient to place the operator to the right of the operand.]

Ex. 12. If $f_n(\rho)$ is any homogeneous function of ρ of the order n which vanishes under the operation of ∇^2, the function $\mathrm{T}\rho^{-2n-1}\,.\,f_n(\rho)$ will vanish under the same operator.

[Expressing that $\nabla^2(\mathrm{T}\rho^m . f_n)=0$, we may write this relation in the form $(\nabla+\nabla')^2 . (\mathrm{T}\rho'^m . f_n)=0$, provided we remove the accents after the operation. This expands into $\nabla'^2\mathrm{T}\rho'^m . f_n+2\mathrm{S}\nabla'\mathrm{T}\rho'^m\nabla . f_n+\mathrm{T}\rho'^m . \nabla^2 f_n=0$, and observing that $\mathrm{S}\rho\nabla . f_n=-nf_n$ because f_n is homogeneous in ρ, we easily find the equation reduces to $m(2n+1+m)=0$. This result is of importance in the theory of spherical harmonics.]

ART. **58**. Given a quaternion function $p=F(q)$ of another quaternion q, we have seen how to express $\mathrm{d}p$ in terms of $\mathrm{d}q$ (Art. 51). It is a more difficult problem to express $\mathrm{d}q$ in terms of $\mathrm{d}p$, and we postpone the general method of solution for the present.* However, there are a few cases in which the problem can be solved directly, such as to find the differential of the square root of a quaternion.

Here $$p=q^{\frac{1}{2}} \quad \text{or} \quad p^2=q, \qquad \text{(I.)}$$
so that $$p\mathrm{d}p+\mathrm{d}p . p=\mathrm{d}q. \qquad \text{(II.)}$$

Multiply this by $\mathrm{K}p$ and into p, and two relations equivalent to (II.) are obtained,

$$\mathrm{K}p . p . \mathrm{d}p+\mathrm{K}p . \mathrm{d}p . p=\mathrm{K}p . \mathrm{d}q ; \quad p . \mathrm{d}p . p+\mathrm{d}p . p^2=\mathrm{d}q . p. \qquad \text{(III.)}$$

Adding, we have

$$\mathrm{d}p . (\mathrm{T}p^2+2p\mathrm{S}p+p^2)=\mathrm{K}p . \mathrm{d}q+\mathrm{d}q . p$$

because $$p+\mathrm{K}p=2\mathrm{S}p, \quad \mathrm{K}p . p=\mathrm{T}p^2 ;$$

or $$4 . \mathrm{d}p . p\mathrm{S}p=\mathrm{K}p . \mathrm{d}q+\mathrm{d}q . p$$

because $\mathrm{T}p^2=(\mathrm{S}p)^2-(\mathrm{V}p)^2$, $p^2=(\mathrm{S}p)^2+2\mathrm{S}p . \mathrm{V}p+(\mathrm{V}p)^2$;

and hence $$\mathrm{d}q^{\frac{1}{2}}=\frac{\mathrm{K}q^{\frac{1}{2}} . \mathrm{d}q . q^{-\frac{1}{2}}+\mathrm{d}q}{4\mathrm{S}q^{\frac{1}{2}}} \qquad \text{(IV.)}$$

As another example, under which this might have been included, to find the differential of the n^{th} root of a quaternion (n being an integer), we have

$$p=q^{\frac{1}{n}}, \quad q=p^n, \quad \mathrm{d}q=\mathrm{d}p . p^{n-1}+p . \mathrm{d}p . p^{n-2}+\ldots+p^{n-1} . \mathrm{d}p. \qquad \text{(V.)}$$

Multiply $\mathrm{d}q$ into p and subtract the product $p\mathrm{d}q$, and

$$\mathrm{d}q . p-p\mathrm{d}q=\mathrm{d}p . q-q\mathrm{d}p, \quad \text{or} \quad \mathrm{V} . \mathrm{V}\mathrm{d}q\mathrm{V}p=\mathrm{V} . \mathrm{V}\mathrm{d}p\mathrm{V}q. \qquad \text{(VI.)}$$

Thus, with an indetermined scalar x,

$$\mathrm{V}\mathrm{d}p=\frac{\mathrm{V} . \mathrm{V}\mathrm{d}q\mathrm{V}p+x}{\mathrm{V}q}, \quad \text{or} \quad \mathrm{d}p=\frac{\mathrm{V} . \mathrm{V}\mathrm{d}q\mathrm{V}p}{\mathrm{V}q}+\left(\mathrm{S}\mathrm{d}p+\frac{x}{\mathrm{V}q}\right). \qquad \text{(VII.)}$$

Turning to (V.), we have on substitution from (VII.),

$$\begin{aligned}\mathrm{d}q&=n . p^{n-1}\mathrm{S}\mathrm{d}p+\mathrm{V}\mathrm{d}p . p^{n-1}+p . \mathrm{V}\mathrm{d}p . p^{n-1}+\ldots+p^{n-1}\mathrm{V}\mathrm{d}p\\ &=n . p^{n-1}\mathrm{S}\mathrm{d}p+\frac{x}{\mathrm{V}q} . np^{n-1}+\frac{\mathrm{V} . \mathrm{V}\mathrm{d}q\mathrm{V}p}{\mathrm{V}q}\\ &\quad\times(p^{n-1}+\mathrm{K}p . p^{n-2}+(\mathrm{K}p)^2 . p^{n-3}+\ldots+(\mathrm{K}p)^{n-1}), \qquad \text{(VIII.)}\end{aligned}$$

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* See Art. 150, p. 273.

because q and p are commutative in order of multiplication, and because $ap = \mathrm{K}p \,.\, a$, or $a(\mathrm{S}p+\mathrm{V}p) = (\mathrm{S}p - \mathrm{V}p)a$ if $\mathrm{S}a\mathrm{V}p = 0$, the vector $\mathrm{V} \,.\, \mathrm{Vd}q\mathrm{V}p \,.\, (\mathrm{V}q)^{-1}$ being perpendicular to $\mathrm{V}p$. Again,

$$p^{n-1} + \mathrm{K}p \,.\, p^{n-2} + \text{etc.} = \frac{p^n - (\mathrm{K}p)^n}{p - \mathrm{K}p} = \frac{\mathrm{V}p^n}{\mathrm{V}p}$$

since p and $\mathrm{K}p$ are commutative in multiplication, and the expression (VIII.) reduces further to

$$\mathrm{d}q = n \,.\, p^{n-1}\mathrm{Sd}p + \frac{x}{\mathrm{V}q} \,.\, np^{n-1} + \frac{\mathrm{V} \,.\, \mathrm{Vd}q\mathrm{V}p}{\mathrm{V}q}\,\frac{\mathrm{V}q}{\mathrm{V}p}. \quad \text{.......(IX.)}$$

Thus we have by (VII.) and (IX.) on elimination of x

$$\mathrm{d}p = \frac{\mathrm{V} \,.\, \mathrm{Vd}q\mathrm{V}p}{\mathrm{V}q}\left(1 - \frac{p\mathrm{V}q}{nq\mathrm{V}p}\right) + \frac{\mathrm{d}q \,.\, p}{nq}, \quad \text{..............(X.)}$$

and the sought differential $\mathrm{d}p$ is expressed in terms of p, q and $\mathrm{d}q$.

ART. **59.** Writing the first differential of fq in the form

$$\mathrm{d} \,.\, fq = f_1(q, \mathrm{d}q), \quad \text{.................................(I.)}$$

to indicate that it is a function of q and of $\mathrm{d}q$, linear in the latter, the second differential may be expressed by

$$\mathrm{d}^2 \,.\, fq = f_2(q, \mathrm{d}q) + f_1(q, \mathrm{d}^2q), \quad \text{..........................(II.)}$$

where $f_2(q, \mathrm{d}q)$ is homogeneous and quadratic in $\mathrm{d}q$.

A similar process holds in general, and in particular if $\mathrm{d}q$ is constant, so that $\mathrm{d}^2q = 0$, $\mathrm{d}^3q = 0$, etc., we have

$$\mathrm{d}^m \,.\, fq = \mathrm{d} \,.\, f_{m-1}(q, \mathrm{d}q) = f_m(q, \mathrm{d}q). \quad \text{.......................(III.)}$$

Suppose that $f(q)$ and its successive differentials up to the m^{th} are finite for finite differentials of q, and consider the function

$$F(x) = f(q+xp) - f(q) - \frac{x}{1} \,.\, f_1(q, p) - \frac{x^2}{1 \,.\, 2} f_2(q, p) \ldots - \frac{x^{m-1}}{\underline{|m-1}} f_{m-1}(q, p), \quad \text{.....(IV.)}$$

in which x is a scalar and q and p are two quaternions. Differentiating with respect to x, and leaving p and q constant, we find by the general relation (III.),

$$\left.\begin{aligned}
\frac{\partial F(x)}{\partial x} &= f_1(q+xp, p) - f_1(q, p) - \frac{x}{1} \,.\, f_2(q, p) \ldots - \frac{x^{m-2}}{\underline{|m-2}} f_{m-1}(q, p), \\
\frac{\partial^2 F(x)}{\partial x^2} &= f_2(q+xp, p) - f_2(q, p) - \frac{x}{1} \,.\, f_3(q, p) \ldots - \frac{x^{m-3}}{\underline{|m-3}} f_{m-1}(q, p), \\
&\vdots \\
\frac{\partial^{m-1} F(x)}{\partial x^{m-1}} &= f_{m-1}(q+xp, p) - f_{m-1}(q, p), \\
\frac{\partial^m F(x)}{\partial x^m} &= f_m(q+xp, p).
\end{aligned}\right\} \ldots\text{(V.)}$$

Putting $x=0$ in (IV.) and (V.), we see that $F(x)$ and its successive deriveds up to the order $m-1$ vanish when $x=0$, and consequently

$$F(x) = \frac{x^m}{\underline{|m}} (f_m(q, p) + r_m), \quad \text{...........................(VI.)}$$

where r_m is some quaternion function of x, q and p, and where by (v.)

$$\lim_{x=0} \frac{F(x)\lfloor m}{x^m} = \lim_{x=0} (f_m(q, p) + r_m) = f_m(q, p). \quad \text{.............(VII.)}$$

By taking x small enough it is consequently possible to render r_m infinitely small in comparison with $f_m(q, p)$, or

$$\lim_{x=0} \frac{\mathrm{T}r_m}{\mathrm{T}f_m(q, p)} = 0. \quad \text{..............................(VIII.)}$$

Replacing xp by p in (IV.), what we have proved is that

$$f(q+p) = f(q) + \frac{1}{1} f_1(q, p) + \frac{1}{1\,.\,2} f_2(q, p) + \ldots + \frac{1}{\lfloor m} \{f_m(q, p) + r_m\}, \quad \text{....(IX.)}$$

where r_m is a function of q and p, which becomes evanescent in comparison with $f_m(q, p)$ for sufficiently small tensors of p. This theorem is what Hamilton calls "Taylor's Series adapted to quaternions."

In certain cases, for a large value of m, the term

$$\frac{1}{\lfloor m} \{f_m(q, p) + r_m\}$$

becomes negligible, and we may write the expansion in the usual symbolic form,

$$f(q+p) = e^{\mathrm{d}} f(q) = f(q) + \frac{1}{1} \cdot f_1(q, p) + \frac{1}{1\,.\,2} f_2(q, p) + \text{etc.}; \quad \mathrm{d}q = p, \quad \text{.........(X.)}$$

or more explicitly for a vector variable, 9

$$f(\rho + \varpi) = e^{-\mathrm{S}\varpi\nabla} \cdot f(\rho) = f(\rho) - \frac{1}{1} \mathrm{S}\varpi\nabla \cdot f\rho + \frac{1}{1\,.\,2} \cdot (\mathrm{S}\varpi\nabla)^2 \cdot f(\rho) + \text{etc.} \quad \text{....(XI.)}$$

ART **60.** Instead of differentiating a second time with the same characteristic d, let the differential of

$$\mathrm{d}f(q) = f_1(q, \mathrm{d}q)$$

be taken for a new characteristic, d′ corresponding to the differentials $\mathrm{d}'q$ and $\mathrm{d}'\mathrm{d}q$ of q and $\mathrm{d}q$. The result may be written

$$\mathrm{d}'\mathrm{d} \,.\, f(q) = f_1(q, \mathrm{d}'\mathrm{d}q) + f_2(q, \mathrm{d}'q, \mathrm{d}q), \quad \text{.....................(I.)}$$

where in full,

$$f_2(q, \mathrm{d}'q, \mathrm{d}q) = \lim_{n=\infty} n \left\{ f_1\left(q + \frac{1}{n}\mathrm{d}'q, \mathrm{d}q\right) - f_1(q, \mathrm{d}q) \right\}. \quad \text{...........(II.)}$$

Reversing the order of differentiation,

$$\mathrm{d}\mathrm{d}' \,.\, f(q) = f_1(q, \mathrm{d}\mathrm{d}'q) + f_2(q, \mathrm{d}q, \mathrm{d}'q). \quad \text{....................(III.)}$$

We shall now prove the relation

$$f_2(q, r, s) = f_2(q, s, r), \quad \text{..............................(IV.)}$$ 9

where r and s are any two quaternions replacing $\mathrm{d}q$ and $\mathrm{d}'q$ in the functions which occur in (I.) and (III.). We have by (II.),

$$\begin{aligned}
f_2(q, r, s) &= \lim_{n=\infty} n \left\{ f_1\left(q + \frac{1}{n} r, s\right) - f_1(q, s) \right\} \\
&= \lim_{n=\infty} \left[\lim_{m=\infty} m \left\{ f\left(q + \frac{1}{n} r + \frac{1}{m} s\right) - f\left(q + \frac{1}{n} r\right) \right\} \right. \\
&\qquad \left. - \lim_{m=\infty} m \left\{ f\left(q + \frac{1}{m} s\right) - f(q) \right\} \right] \\
&= \lim_{n=\infty,\, m=\infty} mn \left\{ f\left(q + \frac{1}{n} r + \frac{1}{m} s\right) - f\left(q + \frac{1}{n} r\right) - f\left(q + \frac{1}{m} s\right) + fq \right\},
\end{aligned}$$

and from symmetry this is equal to $f_2(q, s, r)$.

More generally, *if by successive differentiation of a function* f(q), *a function* $f_n(q, r_1, r_2, \ldots r_n)$ *is constructed, the order in which the quaternions* $r_1, r_2, \ldots r_n$ *are grouped among themselves is immaterial.*

In virtue of (IV.), it appears that

$$d'd \,.\, f(q) - dd' \,.\, f(q) = f_1(q,\, d'dq - dd'q) \,; \qquad \text{(v.)}$$

and in general this difference vanishes if, and only if, $d'dq = dd'q$.

Ex. 1. If Q is a scalar function of ρ, and if $dQ = S\nu d\rho$, $d\nu = \phi d\rho$, show that the function ϕ is self-conjugate, or that $S\alpha\phi\beta = S\beta\phi\alpha$, where α and β are any two vectors.

[This is a particular case of (IV.). Compare Art. 51, Ex. 2, and Art. 62.]

Ex. 2. If ν_1 and ν_2 are any two vector functions of the vector ρ; if $d\nu_1 = \phi_1(d\rho)$ and $d\nu_2 = \phi_2(d\rho)$, and if ∇ operates on all functions of ρ on its right, show that

$$S\nu_1\nabla \,.\, S\nu_2\nabla \,.\, - S\nu_2\nabla \,.\, S\nu_1\nabla \,.\, = S(\phi_1\nu_2 - \phi_2\nu_1)\nabla \,.\,;$$

or in other words prove that the two operators produce the same effect on any function of ρ.

q **Ex. 3.** If p, q and r are any three quantities or operators, not necessarily commutative in order of operation or multiplication, show that

$$[[p,\, q]r] + [[q,\, r]p] + [[r,\, p]q] = 0$$

where $\qquad [p,\, q] = pq - qp, \quad [[p,\, q],\, r] = [p,\, q]r - r[p,\, q].$

Ex. 4. If p and q are any two quantities or operators, show that

q $$e^{-q}pe^{q} = p + \frac{p_1}{1} + \frac{p_2}{1\,.\,2} + \frac{p_3}{1\,.\,2\,.\,3} + \text{etc.}, \quad \text{where} \quad p_n = [p_{n-1},\, q]\,;$$

and hence prove the equation connecting operators,

$$e^{S\nu_1\nabla} S\nu_2\nabla e^{-S\nu_1\nabla} = S\nu_3\nabla,$$

where ν_1 and ν_2 are any given functions of ρ, where ν_3 is a determinate function of ρ and where ∇ operates only on functions on its right.

ART. **61**. To find a stationary value of the scalar function $f(\rho)$, whenever a stationary value exists, we equate to zero the first differential

$$df(\rho) = S\nu d\rho \qquad \text{(I.)}$$

of $f(\rho)$ for all differentials $d\rho$. This requires the vector ν to be zero, for otherwise $S\nu d\rho$ cannot be zero for every differential $d\rho$, and the stationary values are obtained by substituting in $f(\rho)$ the vectors ρ which satisfy the equation

$$\nu = 0. \qquad \text{(II.)}$$

If the stationary value is subject to the condition

$$g(\rho) = 0, \qquad \text{(III.)}$$

where $g(\rho)$ is a given scalar function of ρ, the differential $d\rho$ is no longer arbitrary, and the conditions are

$$df(\rho) = S\nu d\rho = 0, \quad dg(\rho) = S\lambda d\rho = 0, \qquad \text{(IV.)}$$

where λ is a new vector function of ρ defined by the nature of the function $g(\rho)$. Considered geometrically the condition (III.) requires the vector ρ to terminate on a certain surface and constrains the differential $d\rho$ to be tangential to the surface as expressed by $S\lambda d\rho = 0$. The function $f(\rho)$ has a stationary value if $df(\rho)$ vanishes for every differential $d\rho$ at right angles to λ. In other words we must have ν parallel to λ, or

$$\nu + x\lambda = 0, \quad \text{or} \quad V\nu\lambda = 0, \quad \text{.......................(V.)}$$

where x is a scalar multiplier. The solutions of (III.) and (V.) afford vectors ρ which render $f(\rho)$ stationary in value.

Again if there are two equations of condition,

$$g(\rho) = 0, \quad h(\rho) = 0, \quad \text{........................(VI.)}$$

the differential of $d\rho$ consistent with these conditions must satisfy

$$dg(\rho) = S\lambda d\rho = 0, \quad dh(\rho) = S\mu d\rho = 0, \text{............(VII.)}$$

so that $d\rho \parallel V\lambda\mu$, and if in addition $f(\rho)$ is stationary in value so that $df(\rho) = 0$, or $S\nu d\rho = 0$, we must have ν coplanar with λ and μ, or

$$\nu + x\lambda + y\mu = 0, \quad \text{or} \quad S\nu\lambda\mu = 0, \quad \text{................(VIII.)}$$

where x and y are two scalar multipliers. Here the three vanishing scalar functions of ρ, $g(\rho) = 0$, $h(\rho) = 0$ and $S\nu\lambda\mu = 0$, serve to determine a certain number of vectors ρ as vectors to the points of intersection of three known surfaces, and substitution of any one of these vectors in $f(\rho)$ will give a stationary value.

For the solution of the equations, no general rule can be laid down. Sometimes, indeed most frequently, it is more convenient to deal with the equations (V.) and (VIII.) involving x and y rather than with the results of elimination of these scalars.

To examine the nature of the stationary values of $f(\rho)$, it is necessary to proceed to second differentials. For example when there are two equations of condition, we have in addition to (VII.) (compare Art. 51, Ex. 2, Art. 60, Ex. 1),

$$d^2g(\rho) = S\lambda d^2\rho + Sd\rho\phi_{,}d\rho = 0, \quad d^2h(\rho) = S\mu d^2\rho + Sd\rho\phi_{,,}d\rho = 0, \quad \text{(IX.)}$$

where $\phi_{,}$ and $\phi_{,,}$ are two linear vector functions determined by the functions $g(\rho)$ and $h(\rho)$, and we must consider the sign of

$$d^2f(\rho) = S\nu d^2\rho + Sd\rho\phi d\rho, \quad \text{.....................(X.)}$$

when appropriate values of ρ and $d\rho$ are substituted therein.

By adding the equations (IX.) multiplied by x and y to this we have by (VIII.)

$$d^2f(\rho) = Sd\rho(\phi + x\phi_{,} + y\phi_{,,})d\rho, \quad \text{where} \quad d\rho \parallel V\lambda\mu, \text{.......(XI.)}$$

the scalars x and y being given by (VIII.) in terms of ν, λ and μ by means of the relations $V\mu\nu = xV\lambda\mu$, $V\nu\lambda = yV\lambda\mu$, in which we suppose the appropriate value of ρ to be substituted. For the negative sign, $f(\rho)$ is a maximum, while it is a minimum if the sign is positive.

In like manner, when there is only one equation of condition, we find

$$d^2f(\rho) = Sd\rho(\phi + x\phi_{\prime})d\rho, \quad \text{where} \quad S\lambda d\rho = 0, \quad \nu + x\lambda = 0, \quad \text{(XII.)}$$

and if $d^2f(\rho)$ is positive for every $d\rho$ perpendicular to λ the function $f(\rho)$ is a minimum; if $d^2f(\rho)$ changes sign for some vectors $d\rho$ perpendicular to λ, the function is merely stationary; if $d^2f(\rho)$ is constantly negative for the differentials $d\rho$, the function is a maximum.

Ex. 1. Find the stationary values of $T\rho$, subject to the condition,

$$(\rho - \alpha)^2 + a^2 = 0.$$

[Here $dT\rho = -SU\rho d\rho = 0$, where $d\rho$ satisfies $S(\rho - \alpha)d\rho = 0$, so that $U\rho \parallel \rho - \alpha$, or $\rho \parallel \alpha$, or $\rho = x\alpha$ say, and the condition gives

$$(x-1)^2\alpha^2 + a^2 = 0, \quad \text{or} \quad x = 1 \pm aT\alpha^{-1},$$

so that $\rho = \alpha \pm aU\alpha$.]

Ex. 2. Find the stationary values of $T\rho$ when $(\rho - \alpha)^2 + a^2 = 0$, $S\beta\rho = 0$.

EXAMPLES TO CHAPTER VII.

Ex. 1. If $OP = \rho = \alpha^t\beta$, $T\alpha = 1$, $S\alpha\beta = 0$, the locus of the point P will be the circumference of a circle, with O for centre, and OB $(=\beta)$ for radius, and in a plane perpendicular to OA $(=\alpha)$.

Ex. 2. If $OP = \rho = V \,.\, \alpha^t\beta$, $\gamma = OC = V\alpha\beta$, $T\alpha = 1$, the locus of P is an ellipse, with its centre at O, and with OB and OC for its major and minor semiaxes.

Ex. 3. If under the same conditions as in Ex. 2,

$$OB' = \beta' = \alpha^{-1}V\alpha\beta, \quad OP' = \rho' = \alpha^{-1}V\alpha\rho,$$

the locus of P′ is a circle with OB′ and OC for two rectangular radii. The equation of the circle may be written

$$\rho' = \alpha^t\beta'.$$

Ex. 4. If $OP = \rho = a^t\beta$, $S\alpha\beta = 0$, the locus of P is a logarithmic spiral with O for its pole.

Ex. 5. If $OP = \rho = V \,.\, a^t\beta$, the locus of P is an elliptic logarithmic spiral—a plane curve which may be projected into an ordinary logarithmic spiral.

Ex. 6. The equation

$$\rho = ct\alpha + \alpha^t\beta \quad \text{with} \quad S\alpha\beta = 0, \quad T\alpha = 1,$$

represents a helix, while the locus of the perpendiculars to the axis of the helix which intersect the curve is represented by

$$\rho = ct\alpha + u\alpha^t\beta,$$

where u is a variable scalar.*

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* These Examples are taken from the *Elements of Quaternions*, Art. 314.

Ex. 7. If we project the ellipse

$$\rho = \alpha \cos x + \beta \sin x$$

on a plane at right angles to the vector λ, the vectors α and β will project into the principal semiaxes of the projection provided

$$\mathrm{S}\,.\,\mathrm{V}\lambda\alpha\mathrm{V}\lambda\beta = 0.$$

(a) They will project into equi-conjugate radii if

$$\mathrm{TV}\lambda\alpha = \mathrm{TV}\lambda\beta.$$

(b) If $\qquad \mathrm{S}\alpha\mathrm{U}\lambda = \pm\sqrt{[\tfrac{1}{2}(\beta^2-\alpha^2) \pm \sqrt{\{\tfrac{1}{4}(\beta^2-\alpha^2)^2 + (\mathrm{S}\alpha\beta)^2\}}]},$

$$\mathrm{S}\beta\mathrm{U}\lambda = \mp\sqrt{[\tfrac{1}{2}(\alpha^2-\beta^2) \pm \sqrt{\{\tfrac{1}{4}(\beta^2-\alpha^2)^2 + (\mathrm{S}\alpha\beta)^2\}}]},$$

the ellipse will project into a circle—one of four, of which two are imaginary.

(c) The squared radii of the circles of projection are

$$-\tfrac{1}{2}(\alpha^2+\beta^2) \mp \sqrt{\{\tfrac{1}{4}(\beta^2-\alpha^2)^2 + (\mathrm{S}\alpha\beta)^2\}},$$

the upper sign corresponding to the real circles.

Ex. 8. A circle of radius $\pm n^{-1}\mathrm{T}\beta$ rolls on a circle of radius $\mathrm{T}\beta$ and centre O, and carries with it a point P at a distance $l\mathrm{T}\beta$ from its centre. The locus of the point P is represented by

$$\mathrm{OP} = \rho = (1+n^{-1})\alpha^t\beta - l\alpha^{(1+n)t}\beta, \quad \mathrm{T}\alpha = 1, \quad \mathrm{S}\alpha\beta = 0.$$

(a) Prove that $\qquad d\rho = \tfrac{1}{2}\pi(1+n)\alpha(\rho - \alpha^t\beta)dt,$

and assign the geometrical interpretation.

(b) If the variable scalar t represents the time, the equation of the hodograph * is

$$\dot\rho = \tfrac{1}{2}\pi(1+n)\alpha(n^{-1}\alpha^t\beta - l\alpha^{(1+n)t}\beta),$$

and show that this curve may be generated by a point carried by one circle rolling on another.

(c) Show that the condition for a cusp on the path of the point P is

$$1 = nl\alpha^{nt},$$

and discuss fully the nature of this equation.

(d) Prove that the vector of acceleration of the point P for uniform motion of the circle is

$$\ddot\rho = \tfrac{1}{4}\pi^2(1+n)\{(2+n)\alpha^t\beta - (1+n)\rho\},$$

and determine the condition that the acceleration may momentarily vanish.

(e) The condition for an inflexion is found by expressing that $\mathrm{U}d\rho$ is stationary or that $\mathrm{V}d\rho d^2\rho = 0$, and it may be reduced to

$$l^2n^2(1+n) - ln(2+n)\mathrm{S}\alpha^{nt} + 1 = 0.$$

(f) Show that the inflexions lie on the circle

$$\mathrm{T}\rho = \left\{\frac{(3+n)(1+n) - l^2n^2(3+2n)}{n(2+n)}\right\}^{\frac{1}{2}}\mathrm{T}\beta.$$

Ex. 9. Under the same conditions, what curve, or rather what system of curves for various values of the scalar l is represented by $\rho = \beta t + l\alpha^t\beta$?

* The *hodograph* of an orbit is the locus of the extremity of a vector drawn from a fixed point to represent the velocity of the moving body.

Ex. 10. (*a*) If $\text{OQ}=\phi(t)$ and $\text{OQ}'=\psi(u)$ are the equations of any two curves the relation

$$\text{Td}\,.\,\phi(t)=\text{Td}\,.\,\psi(u)$$

is equivalent to a differential equation connecting the parameters so that corresponding values of the parameters in an integral determine equal arcs measured from fixed points on the curves.

(*b*) If the condition (*a*) is satisfied, the quaternion

$$\frac{\text{d}\,.\,\phi(t)}{\text{d}\,.\,\psi(u)}$$

is a versor which renders the tangent to the second curve at u parallel to the tangent to the first curve at the corresponding point t.

(*c*) When the curves lie in a common plane, the condition (*a*) being still satisfied, the equation

$$\text{OP}=\rho=\phi(t)-\frac{\text{d}\,.\,\phi(t)}{\text{d}\,.\,\psi(u)}\psi(u)=\text{OQ}-\text{PQ}$$

is the locus of the pole of the second curve when it rolls along the first so that points answering to corresponding values of the parameters t and u remain in contact.

(*d*) The vector tangential to the roulette at the point P is

$$\text{d}\rho=-\left(\text{d}\,.\,\frac{\text{d}\phi}{\text{d}\psi}\right).\,\psi,$$

and this vector is at right angles to $\rho-\phi(t)$ because the quaternion of (*b*) is a versor.

(*e*) The equation of the normal at the point P is therefore

$$\varpi=\text{OP}+x\text{PQ}=\phi(t)+(x-1)\,.\,\frac{\text{d}\phi}{\text{d}\psi}\,.\,\psi(u).$$

Ex. 11. The earth and a planet being assumed to describe circular orbits round the sun, show that the apparent path of the planet is represented by

$$\rho=\text{U}(c\gamma^{4tP^{-1}}\alpha-b\beta^{4tE^{-1}}\alpha)$$

where c is the radius of the orbit of the planet and b that of the orbit of the earth, where P and E are the periodic times of the planet and the earth, where γ and β are unit vectors normal to the planes of the orbits and where α is a unit vector directed towards a node.

(*a*) Show that the equation

$$\text{S}\beta(c\gamma^{4tP^{-1}}-b\beta^{4tE^{-1}})(cP^{-1}\gamma^{1+4tP^{-1}}-bE^{-1}\beta^{1+4tE^{-1}})=0$$

determines the values of t corresponding to the "stationary points" at which the motion changes from direct to retrograde or *vice versa.*

Ex. 12. Show that the equation

$$\rho=h\text{V}\alpha^{2t}+u\alpha^{t}\beta \quad \text{where} \quad \text{T}\alpha=1, \quad \text{S}\alpha\beta=0$$

represents a cylindroid referred to its centre, and deduce the scalar equation

$$\beta^2\text{V}\alpha\rho^2\text{S}\alpha\rho=2h\text{S}\beta\rho\text{S}\alpha\beta\rho.$$

Ex. 13. Describe the loci represented by the following equations:

(i) $\rho=\alpha\text{S}\lambda\text{U}\tau$;

(ii) $\rho=\alpha\text{S}\lambda\text{U}\tau+\beta\text{S}\mu\text{U}\tau$;

(iii) $\rho=\alpha\text{S}\lambda\text{U}\tau+\beta\text{S}\mu\text{U}\tau+\gamma\text{S}\nu\text{U}\tau$, where α, β, γ, λ, μ and ν are given constant vectors, and when the auxiliary variable vector τ is perfectly arbitrary.

(a) What modifications must be made in your interpretations when τ remains constantly inclined to given direction?

Ex. 14. (a) If $\mathrm{S}\rho d\rho = 0$, show that $\mathrm{T}\rho$ is constant.

(b) If $\mathrm{V}\rho d\rho = 0$, it follows that $\mathrm{U}\rho$ has a fixed direction.

(c) If $\mathrm{S}\rho d\rho d^2\rho = 0$, show that $\mathrm{UV}\rho d\rho$ has a fixed direction and the vectors ρ are parallel to a fixed plane.

Ex. 15. Show that

$$\mathrm{T}(1+q) = (1+q)^{\frac{1}{2}}(1+\mathrm{K}q)^{\frac{1}{2}}, \quad \mathrm{U}(1+q) = (1+q)^{\frac{1}{2}}(1+\mathrm{K}q)^{-\frac{1}{2}}.$$

(*Elements of Quaternions*, Art. 343 (9).)

Ex. 16. Prove the relations

$$\mathrm{U}(\alpha+\beta) = \mathrm{U}\alpha\,.\,(1+\alpha^{-1}\beta)^{\frac{1}{2}}(1+\beta\alpha^{-1})^{-\frac{1}{2}};\ \frac{\mathrm{U}(\alpha+\beta)}{\mathrm{T}(\alpha+\beta)^2} = \frac{\mathrm{U}\alpha}{\mathrm{T}\alpha^2}(1+\alpha^{-1}\beta)^{-\frac{1}{2}}(1+\beta\alpha^{-1})^{-\frac{3}{2}};$$

and find the development to the third order when $\mathrm{T}\beta$ is small in comparison with $\mathrm{T}\alpha$.

Ex. 17. Supposing the earth to describe a circular orbit round the sun, show that the parallactic ellipse of a fixed star is represented by

$$\varpi = -\mathrm{V}\,.\,\gamma^x\alpha\sigma^{-1}\,.\,\mathrm{U}\sigma$$

where σ and $\gamma^x\alpha$ are the heliocentric vectors to the star and to the earth respectively.

(a) Show also that

$$\mathrm{UV}\sigma\gamma\,.\,\mathrm{T}\alpha\sigma^{-1} \quad \text{and} \quad \mathrm{U}\,.\,\sigma\mathrm{V}\sigma\gamma\,.\,\mathrm{T}\alpha\mathrm{S}\gamma\sigma^{-1}$$

are the principal vector radii of the parallactic ellipse.

Ex. 18. If v is the (scalar) velocity of light and $\dot\rho$ the velocity of the earth in its orbit, the aberration of a star is represented by

$$\mathrm{U}(v\mathrm{U}\sigma + \dot\rho) - \mathrm{U}\sigma.$$

(a) The earth's orbit being supposed circular, the aberrational ellipse is given by

$$\varpi = -v^{-1}u\mathrm{VU}\,.\,\gamma^{x+1}\alpha\sigma\,.\,\mathrm{U}\sigma$$

where u is the scalar velocity of the earth.

Ex. 19. Assuming the effect of refraction to be K times the tangent of the zenith distance, show that a star in the direction of the unit vector σ appears to be in the direction of the vector

$$\left(1 + K\frac{\mathrm{V}k\sigma}{\mathrm{S}k\sigma}\right)\sigma$$

where k is the unit vector directed to the zenith.

Ex 20. If P is a point in a body attached at B and C by universal joints to two bars BA and CD having fixed universal joints at A and D, show that the motion of the point P is subject to the conditions implied in the equations

$$\mathrm{AP} = \rho = pap^{-1} + q\epsilon q^{-1}, \quad \mathrm{PD} = \rho' = r\gamma r^{-1} + q\eta q^{-1},$$

where α, γ, ϵ and η are fixed vectors and where p, q and r are variable quaternions; prove that the envelope of the point may be determined by identifying the equations

$$\mathrm{SVd}qq^{-1}\,.\,\mathrm{V}q\epsilon q^{-1}\rho=0,\quad \mathrm{SVd}qq^{-1}\,.\,\mathrm{V}q\eta q^{-1}\rho'=0;$$

and show that these conditions require the five points ABPCD to be coplanar.

Ex. 21. If $\mathrm{S}\sigma\mathrm{d}\rho$ becomes the differential of a scalar function of ρ when multiplied by a suitable factor, show that $\mathrm{S}\sigma\nabla\sigma=0$.

Ex. 22. If $\mathrm{d}\nu$ is the directed element of a surface at the extremity of the vector ρ, the element of solid angle it subtends at the origin is $\mathrm{Sd}\nu\nabla\,.\,\mathrm{T}\rho^{-1}$.

Ex. 23. Show that

$$\mathrm{d}\,.\,e^q=\left(\mathrm{Sd}q+\mathrm{S}\frac{\mathrm{Vd}q}{\mathrm{V}q}\,.\,\mathrm{V}q\right)e^q+\mathrm{V}\,.\,\frac{\mathrm{Vd}q}{\mathrm{V}q}\,.\,\mathrm{V}e^q.$$

Ex. 24. The differential of a function of the vectors ρ and σ, σ being a function of ρ, may be written in the form *

$$\mathrm{d}\,.\,\mathrm{P}=-\mathrm{Sd}\rho(\nabla_\rho-\nabla_\rho'\mathrm{S}\sigma'\nabla_\sigma)\,.\,\mathrm{P}$$

where ∇_ρ and ∇_σ operate respectively on ρ and on σ as explicitly involved in P, and where ∇_ρ' operates on ρ as involved in σ', the accents being removed after the performance of the indicated operations.

(*a*) If P is a *scalar* function of ρ and σ, and if σ is a function of ρ which renders P constant,

$$\nabla_\rho\mathrm{P}-\nabla_\rho'\mathrm{S}\sigma'\nabla_\sigma\mathrm{P}=0.$$

(*b*) If the same function σ renders constant another scalar function Q of ρ and σ, the relation

$$(\mathrm{P},\,\mathrm{Q})=\mathrm{S}\,.\,\mathrm{V}\nabla_\sigma\mathrm{P}\nabla_\sigma\mathrm{Q}\mathrm{V}\nabla_\sigma\quad\text{where}\quad(\mathrm{P},\,\mathrm{Q})=\mathrm{S}(\nabla_\rho\mathrm{P}\nabla_\sigma\mathrm{Q}-\nabla_\rho\mathrm{Q}\nabla_\sigma\mathrm{P})$$

must be satisfied. And if σ can be derived from a scalar function of ρ by the operation of ∇, we must have

$$(\mathrm{P},\,\mathrm{Q})=0.$$

(*c*) If λ_1, μ_1, λ_2 and μ_2 are any vector functions of ρ and σ, the operator

$$\mathrm{S}(\lambda_1\nabla_\rho+\mu_1\nabla_\sigma)\mathrm{S}(\lambda_2\nabla_\rho+\mu_2\nabla_\sigma)-\mathrm{S}(\lambda_2\nabla_\rho+\mu_2\nabla_\sigma)\mathrm{S}(\lambda_1\nabla_\rho+\mu_1\nabla_\sigma)$$

reduces to the form $\mathrm{S}(\lambda_{12}\nabla_\rho+\mu_{12}\nabla_\sigma)$.

(*d*) If P_∇ denotes the operator $\mathrm{S}(\nabla_\rho\mathrm{P}\nabla_\sigma-\nabla_\sigma\mathrm{P}\nabla_\rho)$, we have

$$\mathrm{P}_\nabla\mathrm{Q}=-\mathrm{Q}_\nabla\mathrm{P}=(\mathrm{P},\,\mathrm{Q}),$$

where P and Q are scalar functions; and if R is any third scalar function, the expression

$$\mathrm{P}_\nabla\mathrm{Q}_\nabla\,.\,\mathrm{R}-\mathrm{Q}_\nabla\mathrm{P}_\nabla\,.\,\mathrm{R}=\mathrm{P}_\nabla(\mathrm{Q},\,\mathrm{R})+\mathrm{Q}_\nabla(\mathrm{R},\,\mathrm{P})=(\mathrm{P},\,(\mathrm{Q},\,\mathrm{R}))+(\mathrm{Q},\,(\mathrm{R},\,\mathrm{P}))$$

does not involve the second deriveds of R.

(*e*) Hence $\quad(\mathrm{P},\,(\mathrm{Q},\,\mathrm{R}))+(\mathrm{Q},\,(\mathrm{R},\,\mathrm{P}))+(\mathrm{R},\,(\mathrm{P},\,\mathrm{Q}))\equiv 0$;

and the operator $\quad(\mathrm{P},\,\mathrm{Q})_\nabla=\mathrm{P}_\nabla\mathrm{Q}_\nabla-\mathrm{Q}_\nabla\mathrm{P}_\nabla.$

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* Compare Jacobi's method of solution of partial differential equations and Lie's work on Pfaff's Equation.

Ex. 25. Bright curves are seen on a surface owing to light reflected by scratches on the surface from a source at A to an eye at B. If the scratches are represented by putting $u = const.$ in the equation of the surface $\rho = \phi(t, u)$, show that the equation of the curves may be found by combining the equation of the surface with the result of expressing that

$$\mathrm{T}(\phi - \alpha) + \mathrm{T}(\phi - \beta)$$

is a minimum with respect to t.

(a) If the equation of the surface is $f\rho = 0$ and if $F\rho = u$ is the equation of a family of surfaces through the scratches, the bright curves are given by

$$f\rho = 0, \quad \mathrm{S}\nabla f \nabla F\{\mathrm{U}(\rho - \alpha) + \mathrm{U}(\rho - \beta)\} = 0.$$

(b) The bright lines due to the grooves made in turning a surface of revolution ($\mathrm{T}\rho = f\mathrm{S}k\rho$) lie on the surface

$$\mathrm{S}k\rho\{\mathrm{U}(\rho - \alpha) + \mathrm{U}(\rho - \beta)\} = 0\,;$$

and meridian grooves on the same surface give rise to bright curves on the surface

$$\mathrm{SV}k\rho(\mathrm{U}\rho + kf'\mathrm{S}k\rho)\{\mathrm{U}(\rho - \alpha) + \mathrm{U}(\rho - \beta)\} = 0.$$

Ex. 26. The differential of $\mathrm{T}(\rho - \alpha)$ corresponding to a given differential of ρ ceases to be determinate when ρ comes to coincidence with α unless we know a law according to which ρ tends to coincide with α.

CHAPTER VIII.

LINEAR AND VECTOR FUNCTIONS.

ART. **62.** A vector function of a vector, distributive with respect to that vector, is called a *linear vector function.* Thus if

$$\phi(\alpha+\beta)=\phi\alpha+\phi\beta,\ \ \mathrm{S}\phi\alpha=0,\ \ \mathrm{S}\phi\beta=0, \quad \text{(I.)}$$

for all vectors α and β, the function ϕ is linear and vector. As a corollary to the equations of definition

$$\phi(x\alpha)=x\phi\alpha \quad \text{(II.)}$$

if x is any scalar.

Given the vectors

$$\alpha'=\phi\alpha,\ \ \beta'=\phi\beta,\ \ \gamma'=\phi\gamma, \quad \text{(III.)}$$

the results of operating by ϕ on any three given and non-coplanar vectors, the function ϕ is determinate; for by (I.)

$$\phi\rho=\frac{\alpha'\mathrm{S}\beta\gamma\rho+\beta'\mathrm{S}\gamma\alpha\rho+\gamma'\mathrm{S}\alpha\beta\rho}{\mathrm{S}\alpha\beta\gamma}, \quad \text{(IV.)}$$

since $\rho\mathrm{S}\alpha\beta\gamma=\Sigma\alpha\mathrm{S}\beta\gamma\rho$ for any arbitrary vector ρ.

With a new signification of the vectors, α', β', γ', α, β, γ, any linear function may be reduced to the *trinomial* form,

$$\phi\rho=\alpha'\mathrm{S}\alpha\rho+\beta'\mathrm{S}\beta\rho+\gamma'\mathrm{S}\gamma\rho, \quad \text{(V.)}$$

in which either set of vectors α', β', γ' or α, β, γ may be arbitrarily assumed. For if we resolve $\phi\rho$ along three fixed vectors α', β', γ', the coefficients in the resolution must be scalar and distributive functions of ρ; that is, they must be of the form $\mathrm{S}\alpha\rho$, $\mathrm{S}\beta\rho$, $\mathrm{S}\gamma\rho$. If, on the other hand, we assume α, β and γ, the set α', β' and γ' follow, being $\phi\mathrm{V}\beta\gamma:\mathrm{S}\alpha\beta\gamma$, etc.

Thus in any case, the general linear function is seen to involve nine constants, the nine constituents of three vectors α, β and γ, or α', β' and γ'.

For arbitrary vectors, α and β, if

$$\mathrm{S}\alpha\phi\beta=\mathrm{S}\beta\phi'\alpha, \quad \text{(VI.)}$$

the function ϕ' is said to be the *conjugate* of the function ϕ. The conjugate for the trinomial form (V.) is

$$\phi'\rho = \alpha S\alpha'\rho + \beta S\beta'\rho + \gamma S\gamma'\rho. \quad \text{(VII.)}$$

Ex. 1. Given $\sigma = \phi\rho = \alpha' S\alpha\rho + \beta' S\beta\rho + \gamma' S\gamma\rho$,
show that

$$\rho = \phi^{-1}\sigma = (V\beta\gamma S\beta'\gamma'\sigma + V\gamma\alpha S\gamma'\alpha'\sigma + V\alpha\beta S\alpha'\beta'\sigma) : (S\alpha'\beta'\gamma' S\alpha\beta\gamma).$$

Ex. 2. Show that $V\alpha\rho\beta$ is a linear vector function of ρ, and find its conjugate.

Ex. 3. Is $\alpha T\rho$ a linear vector function of ρ?

ART. **63.** From a geometrical point of view the equation

$$\sigma = \phi\rho, \quad \text{(I.)}$$

in which ϕ is a given linear and vector function, and in which the vector ρ is arbitrary, establishes a linear transformation from vectors ρ to vectors σ.

Equal vectors are converted by ϕ into equal vectors; right lines transform into right lines, and planes into planes, as expressed by the relations

$$\sigma = \phi\alpha + t\phi\beta \text{ if } \rho = \alpha + t\beta;$$

$$\sigma = \phi\alpha + t\phi\beta + u\phi\gamma \text{ if } \rho = \alpha + t\beta + u\gamma \quad \text{(II.)}$$

—consequences of the formula of definition (Art. 62 (I.)).

The plane whose equation is

$$S(\rho - \alpha)\beta\gamma = 0 \text{ becomes } S(\sigma - \phi\alpha)\phi\beta\phi\gamma = 0; \quad \text{(III.)}$$

and the vector area

$$V\alpha\beta \text{ transforms into } V\phi\alpha\phi\beta; \quad \text{(IV.)}$$

while the volume

$$S\alpha\beta\gamma \text{ becomes } S\phi\alpha\phi\beta\phi\gamma. \quad \text{(V.)}$$

Ex. 1. Verify that

$$\frac{S\phi\alpha\phi\beta\phi\gamma}{S\alpha\beta\gamma} = \frac{S\phi\alpha'\phi\beta'\phi\gamma'}{S\alpha'\beta'\gamma'} (= m),$$

where α, β, γ and α', β', γ' are any two sets of non-coplanar vectors.

Ex. 2. Prove that

$$V\phi\alpha\phi\beta + V\phi\gamma\phi\delta = V\phi\epsilon\phi\zeta, \text{ if } V\alpha\beta + V\gamma\delta = V\epsilon\zeta.$$

[Take α' along the edge of the planes of $\alpha\beta$ and of $\gamma\delta$, and reduce $V\alpha\beta$ and $V\gamma\delta$ to $V\alpha'\beta'$ and $V\alpha'\gamma'$, etc.]

Ex. 3. Prove that $V\phi\alpha\phi\beta$ is a linear vector function of $V\alpha\beta$.

[This is practically included in the last example. Verify by the trinomial form.]

ART. **64**. There is an inverse transformation which converts vectors σ into vectors ρ, so that

$$\rho = \phi^{-1}\sigma \text{ if } \sigma = \phi\rho; \quad \text{......................(I.)}$$

and we propose to investigate this transformation.

Writing
$$\phi\rho = \sigma = \mathrm{V}\lambda\mu, \quad \text{..........................(II.)}$$

the conditions of perpendicularity of the vectors σ, λ and σ, μ give

$$\mathrm{S}\lambda\phi\rho = 0, \quad \mathrm{S}\mu\phi\rho = 0, \quad \text{or} \quad \mathrm{S}\rho\phi'\lambda = 0, \quad \mathrm{S}\rho\phi'\mu = 0 \quad \text{......(III.)}$$

by the property of the conjugate function (Art. 62 (VI.)).

Thus the vectors $\phi'\lambda$ and $\phi'\mu$ are at right angles to ρ, and consequently

$$m\rho = \mathrm{V}\phi'\lambda\phi'\mu = \psi\mathrm{V}\lambda\mu, \quad \text{or} \quad m\rho = \psi\sigma, \quad \text{............(IV.)}$$

ψ being an auxiliary linear and vector function defined by the equation

$$\psi\mathrm{V}\alpha\beta = \mathrm{V}\phi'\alpha\phi'\beta, \quad \text{........................(V.)}$$

in which α and β are any arbitrary vectors. (See the last Article and its Examples.)

To determine the value of the scalar m operate on (IV.) by $\mathrm{S}\phi'\nu$, where ν is an arbitrary vector, and we have

$$m\mathrm{S}\lambda\mu\nu = \mathrm{S}\phi'\lambda\phi'\mu\phi'\nu, \quad \text{.......................(VI.)}$$

because
$$\mathrm{S}\rho\phi'\nu = \mathrm{S}\nu\phi\rho = \mathrm{S}\nu\sigma = \mathrm{S}\nu\lambda\mu.$$

Operating likewise on (IV.) by ϕ, we have

$$m\phi\rho = \phi\psi\sigma \quad \text{or} \quad m\sigma = \phi\psi\sigma;$$

and replacing σ by $\phi\rho$ we also find

$$m\rho = \psi\phi\rho;$$

so that we may write symbolically

$$m = \phi\psi = \psi\phi, \quad \text{......................(VII.)}$$

with the interpretation that the effect of operating first by ψ and then by ϕ on any vector, or first by ϕ and then by ψ, is to multiply that vector by the scalar m. This relation shows that m is an invariant, or absolutely independent of any particular set of vectors λ, μ, ν in (VI.), for by (V.) ψ is independent of the vectors λ and μ in (IV.). (See also Ex. 1, Art. 63.)

Thus wherever m is not zero, we can always pass from vectors σ to vectors ρ by the relation

$$m\rho = \psi\sigma, \quad \text{..........................(VIII.)}$$

m being calculated by (VI.) and ψ by (V.); and it will be observed that in the calculation of this scalar and this auxiliary function, we only require the direct operation of the function ϕ' on vectors.

Ex. 1. Show that the function ψ transforms vector areas into vector areas when vectors are transformed by the function ϕ'.

Ex. 2. Show that volumes are altered in the ratio $m:1$ in transformation by the function ϕ'.

Ex. 3. Show that ψ' is the conjugate of ψ if $\psi'\mathrm{V}\alpha\beta = \mathrm{V}\phi\alpha\phi\beta$.

[Expand $\mathrm{SV}\gamma\delta\mathrm{V}\phi\alpha\phi\beta$, and prove that it is equal to $\mathrm{SV}\alpha\beta\mathrm{V}\phi'\gamma\phi'\delta$.]

Ex. 4. Show that volumes are altered in the ratio $m:1$ by the transformation produced by ϕ.

[$m\mathrm{S}\alpha\beta = \mathrm{S}\alpha\phi\psi\beta = \mathrm{S}\phi'\alpha\psi\beta = \mathrm{S}\,.\,\psi'\phi'\alpha\beta$.]

Ex. 5. Follow in detail the geometrical meaning of the transformation employed in deducing

$$m\rho = \psi\sigma \quad \text{from} \quad \sigma = \phi\rho.$$

[See Art. 63 (IV.) and Art. 150.]

ART. **65.** The transformation in the last article fails in one case—if m is zero. In that case the vectors σ are all coplanar, the volume of any parallelepiped formed by them being zero (Ex. 4, Art. 64); and because in general $m\rho = \psi\sigma$ if $\sigma = \phi\rho$, in this particular case, the function ψ destroys every vector in the plane. 9

To cover this case, consider the general transformation for an arbitrary function ϕ,

$$\sigma = (\phi + c)\rho = \phi_c\rho \quad \text{and} \quad m_c\rho = \psi_c\sigma, \quad \text{.................(I.)}$$

where c is a scalar and where m_c and ψ_c bear the same relation to $\phi + c$ that m and ψ bear to ϕ. It appears at once by (V.) and (VI.), Art. 64, that

$$m_c\mathrm{S}\lambda\mu\nu = \mathrm{S}(\phi' + c)\lambda(\phi' + c)\mu(\phi' + c)\nu,$$
$$\psi_c\mathrm{V}\lambda\mu = \mathrm{V}(\phi' + c)\lambda(\phi' + c)\mu; \quad \text{.......................(II.)}$$

so that if we write

$$m_c = m + m'c + m''c^2 + c^3, \quad \psi_c = \psi + c\chi + c^2, \quad \text{.........(III.)}$$

we shall have

$$\left.\begin{aligned} m'\mathrm{S}\lambda\mu\nu &= \mathrm{S}\lambda\phi'\mu\phi'\nu + \mathrm{S}\phi'\lambda\mu\phi'\nu + \mathrm{S}\phi'\lambda\phi'\mu\nu, \\ m''\mathrm{S}\lambda\mu\nu &= \mathrm{S}\phi'\lambda\mu\nu + \mathrm{S}\lambda\phi'\mu\nu + \mathrm{S}\lambda\mu\phi'\nu, \\ \chi\mathrm{V}\lambda\mu &= \mathrm{V}\phi'\lambda\mu + \mathrm{V}\lambda\phi'\mu. \end{aligned}\right\} \quad \text{.........(IV.)}$$

Now for any arbitrary value of the scalar c, the scalar m_c is an invariant, and therefore, separately, the coefficients in its expansion m, m' and m'' are invariants, or are independent of λ, μ and ν.

By (I.) we have identically for all scalars c,

$$m_c = \phi_c\psi_c = \psi_c\phi_c, \quad \text{...........................(V.)}$$

or $$m+m'c+m''c^2+c^3=(\phi+c)(\psi+c\chi+c^2)$$
$$=\phi\psi+c(\psi+\phi\chi)+c^2(\phi+\chi)+c^3$$
$$=(\psi+c\chi+c^2)(\phi+c)$$
$$=\psi\phi+c(\psi+\chi\phi)+c^2(\chi+\phi)+c^3,$$
and therefore equating the coefficients of c on each side
$$m=\phi\psi=\psi\phi;\quad m'=\psi+\phi\chi=\psi+\chi\phi;\quad m''=\phi+\chi;\quad \text{...(VI.)}$$
it being understood that these equations denote that equal results are obtained by operating with right or left hand numbers on an arbitrary vector.

9 One of the transformations most frequently required in quaternions is to invert a function $\phi+c$, or to replace an equation $\sigma=(\phi+c)\rho$ by $m_c\rho=\psi_c\sigma$; and in general the process, due to Hamilton, as given in the text is the shortest and most certain. We first calculate $\mathrm{V}(\phi'+c)\lambda(\phi'+c)\mu$ and express it in terms of $\mathrm{V}\lambda\mu$. Then we either calculate m_c from (II.), or it is sometimes better to calculate it directly from (V.), namely from
$$m_c\mathrm{V}\lambda\mu=\phi_c\psi_c\mathrm{V}\lambda\mu.$$
In particular
$$m\rho=\psi\sigma,\quad m'\rho=\psi\rho+\chi\sigma,\quad m''\rho=\chi\rho+\sigma \text{ if } \sigma=\phi\rho;\quad \text{...(VII.)}$$
and thus the general solution of $\sigma=\phi\rho$ is $m'\rho=\chi\sigma+\psi\rho$ if m is zero with the implied condition $\psi\sigma=0$; while if $m=m'=0$, the general solution is $m''\rho=\sigma+\chi\rho$ with the implied conditions $\psi\sigma=0$, $\psi\rho+\chi\sigma=0$. In the first case ($m=0$, $m'\neq 0$), the vector ρ may be considered arbitrary in $\psi\rho$—there is in fact nothing to determine it. But as ψ destroys every vector in the plane of the vectors σ, it is really only the component of the vector normal to that plane that is of any account in $\psi\rho$. In the second case ($m=m'=0$), similar remarks apply; the vector ρ is arbitrary on the right subject to the condition $\psi\rho+\chi\sigma=0$. The function ψ may vanish identically, and this case we shall consider in Art. 66.

Ex. 1. Determine the functions m, ψ and χ for the function $\phi\rho=\Sigma\alpha'\mathrm{S}\alpha\rho$.
[$\psi\rho=\Sigma\mathrm{V}\alpha\beta\mathrm{S}\beta'\alpha'\rho$; $\chi\rho=\Sigma\mathrm{V}\alpha\mathrm{V}\alpha'\rho$; $m\rho=\phi\psi\rho=\Sigma\mathrm{S}\alpha\beta\gamma\mathrm{S}\gamma'\beta'\alpha'\,.\,\rho$.]

Ex. 2. Find the auxiliary functions for $\phi\rho=\mathrm{V}\lambda\rho\mu$.
[Find ϕ_c and ψ_c for $\lambda\mathrm{S}\mu\rho+\mu\mathrm{S}\rho\lambda=\phi_o\rho$.]

Ex. 3. Solve the equations $\sigma=\mathrm{V}\alpha\mathrm{V}\beta\rho$ and $\sigma=\mathrm{V}\alpha\rho$ by the general method, and directly.

Ex. 4. Express $\psi_{c'}$ and $\chi_{c'}$ in terms of ψ_c and χ_c.

Ex. 5. Construct a linear vector function which renders four given vectors parallel to four others.
[The data are $\phi\alpha\,||\,\alpha'$, $\phi\beta\,||\,\beta'$, $\phi\gamma\,||\,\gamma'$, $\phi\delta\,||\,\delta'$, and the function is
$$\phi\rho=c\,.\left(\frac{\alpha'\mathrm{S}\beta'\gamma'\delta'}{\mathrm{S}\beta\gamma\delta}\mathrm{S}\beta\gamma\rho+\frac{\beta'\mathrm{S}\gamma'\alpha'\delta'}{\mathrm{S}\gamma\alpha\delta}\mathrm{S}\gamma\alpha\rho+\frac{\gamma'\mathrm{S}\alpha'\beta'\delta'}{\mathrm{S}\alpha\beta\delta}\mathrm{S}\alpha\beta\rho\right),$$
where c is arbitrary.]

Ex. 6. Prove that

$$m'\mathrm{V}\alpha\beta=\mathrm{V}\phi'\alpha\phi'\beta+\phi\mathrm{V}\phi'\alpha\beta+\phi\mathrm{V}\alpha\phi'\beta\,;$$
$$m''\mathrm{V}\alpha\beta=\mathrm{V}\phi'\alpha\beta+\mathrm{V}\alpha\phi'\beta+\phi\mathrm{V}\alpha\beta.$$

[See equation (VI.). These relations are often useful.]

Ex. 7. Prove that

$$m\phi'\mathrm{V}\alpha\beta=\mathrm{V}\psi\alpha\psi\beta\,;\quad \psi\mathrm{V}\alpha\psi'\beta=m\mathrm{V}\phi'\alpha\beta\,;\quad \phi\mathrm{V}\phi'\alpha\beta=\mathrm{V}\alpha\psi'\beta.$$

Ex. 8. Prove that the equation

$$\rho=(\phi+t)^{-1}\alpha,\quad \text{or}\quad \mathrm{V}(\phi\rho-\alpha)\rho=0,$$

α being a fixed vector and t a variable scalar, represents a twisted cubic.

[Show that it cuts an arbitrary plane in three points.]

Art. **66**. From the equations of the last article connecting ϕ, χ and ψ we deduce

$$\chi=m''-\phi\,;\quad \psi=m'-m''\phi+\phi^2\,;\quad 0=m-m'\phi+m''\phi^2-\phi^3\,;\quad \ldots\text{(I.)}$$

and we have the corresponding equations for the conjugate ϕ',

$$\chi'=m''-\phi'\,;\quad \psi'=m'-m''\phi'+\phi'^2\,;\quad 0=m-m'\phi'+m''\phi'^2-\phi'^3.\ \text{(II.)}$$

These may be proved by reflecting that

$$\mathrm{S}\alpha\phi^2\beta=\mathrm{S}\phi'\alpha\phi\beta=\mathrm{S}\beta\phi'^2\alpha,\ \text{etc.}\,;$$

so that for example

$$\mathrm{S}\alpha\chi\beta=\mathrm{S}\alpha(m''-\phi)\beta=\mathrm{S}\beta(m''-\phi')\alpha=\mathrm{S}\beta\chi'\alpha,$$

and from the third and fourth of these we have $(m''-\phi')\alpha=\chi'\alpha$ because β is perfectly arbitrary.

Let g_1, g_2 and g_3 be the roots of the scalar cubic,

$$0=m-m'g+m''g^2-g^3=0\,;\ \ldots\ldots\ldots\ldots\ldots\text{(III.)}$$

so that $m=g_1g_2g_3$, $m'=g_2g_3+g_3g_1+g_1g_2$, $m''=g_1+g_2+g_3$. ...(IV.)

This scalar cubic is called the *latent* cubic of the function, and its roots are the latent roots of the function ϕ.

We may now write the symbolic cubic (I.) satisfied by the function ϕ in the form

$$(\phi-g_1)(\phi-g_2)(\phi-g_3)=0,\ \ldots\ldots\ldots\ldots\ldots\text{(V.)}$$

and the same symbolic cubic is satisfied by ϕ'. Hence

$$\mathrm{S}(\phi'-g_1)\alpha\,.\,(\phi-g_2)(\phi-g_3)\beta=\mathrm{S}\alpha(\phi-g_1)(\phi-g_2)(\phi-g_3)\beta=0\ \text{(VI.)}$$

whatever vectors α and β may be; or in other words the vector $(\phi'-g_1)\alpha$ is perpendicular to the vector $(\phi-g_2)(\phi-g_3)\beta$. The vectors α and β being both arbitrary, it follows that one or other of the vectors $(\phi'-g_1)\alpha$ or $(\phi-g_2)(\phi-g_3)\beta$ must be parallel to a fixed direction.

But $(\phi'-g_1)\alpha$ is not generally parallel to a fixed direction when the vector α is arbitrary, for if it were we should have

$$\mathrm{V}(\phi'-g_1)\alpha(\phi'-g_1)\beta=0\ \text{ or }\ (\psi-g_1\chi+g_1{}^2)\mathrm{V}\alpha\beta=0,$$

where α and β are quite arbitrary; or symbolically,

$$\psi-g_1\chi+g_1^2=0, \text{ or } (\phi-g_2)(\phi-g_3)=0, \quad\text{.........(VII.)}$$

utilizing (I.) and (IV.), and replacing $m''-g_1$ and $m'-g_1m''+g_1^2$ by their values, g_2+g_3 and g_2g_3. In this case, which is quite special, the symbolic cubic of the function degrades into a quadratic (VII.).

We conclude therefore that the product of a pair of factors of (V.) operating on an arbitrary vector reduces it to a fixed direction, and writing

$$(\phi-g_2)(\phi-g_3)\rho \parallel \gamma_1; \quad (\phi-g_3)(\phi-g_1)\rho \parallel \gamma_2;$$
$$(\phi-g_1)(\phi-g_2)\rho \parallel \gamma_3 \quad\text{.....................(VIII.)}$$

the directions of the vectors γ_1, γ_2, γ_3 are fixed and are called the *axes* of the function ϕ.

We have by (V.),

$$\phi\gamma_1=g_1\gamma_1, \quad \phi\gamma_2=g_2\gamma_2, \quad \phi\gamma_3=g_3\gamma_3; \quad\text{.............(IX.)}$$

and these vectors are generally distinct if the latent roots g_1, g_2, g_3 are unequal, and they are also generally non-coplanar. Resolving then any vector ρ along γ_1, γ_2 and γ_3 we have

$$\rho=x\gamma_1+y\gamma_2+z\gamma_3; \quad\text{.....................(X.)}$$
$$(\phi-g_1)\rho=y(g_2-g_1)\gamma_2+z(g_3-g_1)\gamma_3;$$
$$(\phi-g_2)\rho=x(g_1-g_2)\gamma_1+z(g_3-g_2)\gamma_3;$$
$$(\phi-g_3)\rho=x(g_1-g_3)\gamma_1+y(g_2-g_3)\gamma_2;$$
$$(\phi-g_2)(\phi-g_3)\rho=x(g_1-g_2)(g_1-g_3)\gamma_1;$$
$$(\phi-g_3)(\phi-g_1)\rho=y(g_2-g_3)(g_2-g_1)\gamma_2;$$
$$(\phi-g_1)(\phi-g_2)\rho=z(g_3-g_1)(g_3-g_2)\gamma_3.$$

Thus $(\phi-g_1)\rho$ is coplanar with the pair of axes γ_2 and γ_3, and if γ_1' is the axis of the conjugate function corresponding to the root g_1, it follows from the equation

$$S\rho(\phi'-g_1)\gamma_1'=0=S\gamma_1'(\phi-g_1)\rho \quad\text{................(XI.)}$$

that the vector γ_1' is perpendicular to the plane of $(\phi-g_1)\rho$, and in particular to the vectors γ_2 and γ_3. If vectors are drawn from the centre of a sphere along the axes of a function and of its conjugate, the two spherical triangles the two sets of axes determine are supplemental.

Conceive the function ϕ to undergo continuous variation so that two latent roots, g_2 and g_3, approach coincidence. The corresponding axes approach and ultimately coincide, but their plane is still determinate being perpendicular to γ_1'. Similarly all three axes may coincide in a line perpendicular to that in which the three axes of the conjugate simultaneously coincide.

We shall give an illustration of a function having three equal roots. Let $\phi\alpha=\beta$, $\phi\beta=\gamma$, $\phi\gamma=0$, then $\phi^2\beta=0$, $\phi^3\alpha=0$ and generally $\phi^3\rho=0$, but $\phi^2\rho$ and $\phi\rho$ are not zero. The function is

$$\phi\rho=(\beta S\beta\gamma\rho+\gamma S\gamma\alpha\rho) : S\alpha\beta\gamma, \text{ and } \phi^2\rho=\gamma S\beta\gamma\rho : S\alpha\beta\gamma=\psi\rho.$$

A totally different class of functions is characterized by the equivalent conditions that the axes are indeterminate or that the function satisfies a symbolic quadratic and not a cubic (compare (VII.)). If γ_2 and γ_3 are two different axes corresponding to the same root g_2, the function $\phi-g_2$ destroys every vector in the plane of γ_2 and γ_3, and the function is of the form

$$\phi\rho=g_2\rho+(g_1-g_2)\gamma_1 S\gamma_2\gamma_3\rho : S\gamma_1\gamma_2\gamma_3;$$

and $$(\phi-g_1)(\phi-g_2)=0. \quad\text{.........................(XII.)}$$

The latent cubic has two roots equal to g_2 and the third equal to g_1.

Finally a third class may be noticed—that for which three non-coplanar axes answer to the same root—but a function of this kind is simply a scalar constant.

In general the latent roots may all be real, or two may be imaginary. Corresponding to imaginary roots $g_2=g+\sqrt{-1}\,g'$ and $g_3=g-\sqrt{-1}\,g'$, the axes must be of the form $\gamma_2=\gamma+\sqrt{-1}\,\gamma'$ and $\gamma_3=\gamma-\sqrt{-1}\,\gamma'$. For $(\phi-g_1)[(\phi-g_2)+(\phi-g_3)]$ is real and must produce a real vector from a real vector; but

$$(\phi-g_1)[(\phi-g_2)-(\phi-g_3)]$$

is imaginary and produces an imaginary vector from a real vector.

Ex. 1. Every function coaxial with a given function ϕ is of the form

$$x\psi+y\chi+z.$$

[If h_1, h_2 and h_3 are assumed to be the three roots of the function—the only disposable constants—we find on operating by $x\psi+y\chi+z$ on γ_1, γ_2 and γ_3, three equations which determine x, y and z.]

Ex. 2. Coaxial linear functions are commutative in order of operation, and conversely functions that are commutative are coaxial.

[The first part is easily proved on expressing an arbitrary vector in terms of the axes. The second part is established by operating on the axes. Of course one function may have indeterminate axes. If so, two axes of the other must lie in their plane.]

Ex. 3. Find the latent and symbolic cubics for ψ and χ.

Ex. 4. The equation

$$S\rho\phi\rho\psi\rho=0, \text{ or } S\rho\phi\rho\phi^2\rho=0,$$

represents the three planes through pairs of axes of ϕ.

Ex. 5. In general, if $(\phi^2+x.\phi+y)\rho=0$,

where x and y are scalars, and ρ any given vector, either ρ must be an axis, and the corresponding root must satisfy the quadratic

$$g^2+xg+y=0,$$

or else ρ must be coplanar with a pair of axes, and the corresponding roots must both satisfy the quadratic.

Ex. 6. Deduce the symbolic cubic from the result of replacing λ, μ and ν by $\phi\rho$, $\phi^2\rho$ and $\phi^3\rho$ in the relation

$$\rho S\lambda\mu\nu=\lambda S\mu\nu\rho+\mu S\nu\lambda\rho+\nu S\lambda\mu\rho.$$

ART. **67**. Combining a function and its conjugate by way of addition and subtraction we obtain two more functions,

$$\Phi\rho=\tfrac{1}{2}(\phi+\phi')\rho \text{ and } V\epsilon\rho=\tfrac{1}{2}(\phi-\phi')\rho. \quad \text{.........(I.)}$$

To justify the form attributed to the second function, observe that

$$S\rho(\phi-\phi')\rho=0 \quad \text{.........................(II.)}$$

whatever vector ρ may be.

The function Φ is said to be self-conjugate. The conjugate of $V\epsilon\rho$ is $-V\epsilon\rho$, and the vector ϵ has been called the spin-vector of ϕ.

The axes of a self-conjugate function are mutually rectangular. The function being its own conjugate, each axis must be perpendicular to the other two. *The axes of a real self-conjugate function must be real.* If two are imaginary they must be of the form $\gamma+\sqrt{-1}\gamma'$ and $\gamma-\sqrt{-1}\gamma'$ by the last article, and the condition of perpendicularity requires

$$S(\gamma+\sqrt{-1}\gamma')(\gamma-\sqrt{-1}\gamma')=\gamma^2+\gamma'^2=0,$$

which cannot be, as γ^2 and γ'^2 are both negative. Hence follows the important proposition that *the latent roots of a real self-conjugate function are real.*

If two roots of a real self-conjugate function are equal, it must have indeterminate axes. For if a single axis corresponds to the double root, it must be perpendicular to itself, and therefore imaginary.

Referred to the axes a self-conjugate function is of the form

$$\phi\rho=-g_1 iSi\rho-g_2 jSj\rho-g_3 kSk\rho, \quad \text{...............(III.)}$$

and the only special case is when two of the roots become equal.

An arbitrary self-conjugate function involves only six constants; the three roots and three numbers to fix the directions of the axes.

Ex. 1. The axes of $V\epsilon\rho$ are ϵ and $\epsilon'\pm\sqrt{-1}\epsilon''$, where ϵ, ϵ' and ϵ'' are mutually rectangular, and where $T\epsilon'=T\epsilon''$.

[Note that $(\epsilon'+\sqrt{-1}\epsilon'')^2=0$. The imaginary axes are the vectors to the circular points in the plane $S\epsilon\rho=0$. See Art. 84, Ex. 8, p. 126.]

Ex. 2. Find the self-conjugate part of the function

$$\phi\rho = \alpha' S\alpha\rho + \beta' S\beta\rho + \gamma' S\gamma\rho,$$

and also its spin-vector.

Ex 3. If a self-conjugate function transforms a given vector α into a given vector α', it transforms any other vector $\beta(=\text{OB})$ into a vector, $\beta'(=\text{OB}')$ terminating on a fixed plane.

[Here $S\alpha\beta' = S\alpha'\beta$, and α, α' and β are given.]

Ex. 4. Given that a self-conjugate function renders α parallel to α' and β parallel to β', it renders γ parallel to a fixed plane.

[The conditions of self-conjugation require $S\beta\gamma' S\gamma\alpha' S\alpha\beta' = S\gamma\beta' S\alpha\gamma' S\beta\alpha'$.]

Ex. 5. The axes of a function are mutually rectangular. It is self-conjugate.

Ex. 6. Two axes of a function are at right angles. The spin-vector lies in their plane.

[$S\gamma_1\gamma_2 = 0$, $S\gamma_1\phi\gamma_2 = 0 = S\gamma_2\phi'\gamma_1 = S\gamma_2(\phi - 2\epsilon)\gamma_1$, etc.]

Ex. 7. Prove that the quaternions

$$q_1 = (\phi\lambda \,.\, V\mu\nu + \phi\mu \,.\, V\nu\lambda + \phi\nu \,.\, V\lambda\mu) : S\lambda\mu\nu,$$

$$q_2 = (\lambda \,.\, V\phi\mu\phi\nu + \mu \,.\, V\phi\nu\phi\lambda + \nu \,.\, V\phi\lambda\phi\mu) : S\lambda\mu\nu,$$

are invariants.

[Verify that $q_1 = m'' + 2\epsilon$, $q_2 = m' - 2\phi\epsilon$.]

Ex. 8. If the vectors α, β and γ are mutually perpendicular,

$$V\alpha^{-1}\phi\alpha + V\beta^{-1}\phi\beta + V\gamma^{-1}\phi\gamma = 0,$$

when ϕ is self-conjugate.

Ex. 9. The planes containing a pair of axes of a function and the corresponding pair of axes of its conjugate intersect in the vector $(\phi - g)\epsilon$, where ϵ is the spin-vector and g is a latent root.

Ex. 10. The vector to the common orthocentre of the spherical triangles determined by the axes of a function and its conjugate is

$$UV\epsilon\phi\epsilon.$$

Ex. 11. The spin-vectors of coaxial functions lie in a fixed plane.

Ex. 12. In terms of the roots and axes

$$2\epsilon S\gamma_1\gamma_2\gamma_3 = (g_2 - g_3)\gamma_1 S\gamma_2\gamma_3 + (g_3 - g_1)\gamma_2 S\gamma_3\gamma_1 + (g_1 - g_2)\gamma_3 S\gamma_1\gamma_2.$$

ART. **68.** It happens not unfrequently to be necessary to discriminate between the parts of ψ, χ, and of the invariants which arise from the self-conjugate part of ϕ and those which depend on ϵ. We have

$$\begin{aligned}\psi V\lambda\mu &= V(\Phi - V\epsilon)\lambda(\Phi - V\epsilon)\mu \\ &= \Psi V\lambda\mu - V \,.\, V\epsilon\lambda \,.\, \Phi\mu - V\Phi\lambda V\epsilon\mu + V \,.\, V\epsilon\lambda V\epsilon\mu \\ &= \Psi V\lambda\mu + \lambda S\epsilon\Phi\mu - \mu S\epsilon\Phi\lambda - \epsilon S\epsilon\lambda\mu,\end{aligned}$$

the terms $-\epsilon S\lambda\Phi\mu + \epsilon S\mu\Phi\lambda$ cancelling.

This easily reduces to

$$\psi\rho = \Psi\rho - V\Phi\epsilon\rho - \epsilon S\epsilon\rho. \quad \text{......................(I.)}$$

Thus the spin-vector of ψ is $-\Phi\epsilon$ or $-\phi\epsilon$.

Operating by ϕ or $\Phi + V\epsilon$ we have

$$m\rho = M\rho + V\epsilon\Psi\rho - \Phi V\Phi\epsilon\rho - V\epsilon V\Phi\epsilon\rho - \Phi\epsilon S\epsilon\rho,$$

and if we notice that $\Phi V\Phi\epsilon\rho = V\epsilon\Psi\rho$ (Ex. 7, Art. 65), this reduces without trouble to

$$m = M - S\epsilon\Phi\epsilon \quad \text{or} \quad M = m + S\epsilon\phi\epsilon \quad \text{................(II.)}$$

where M is an invariant of Φ. Changing ϕ into $\phi + c$, and therefore m into $m + m'c + m''c^2 + c^3$, Φ into $\Phi + c$ and M into $M + M'c + M''c^2 + c^3$, we see by (II.) that

$$m' = M' - \epsilon^2 \quad \text{or} \quad M' = m' + \epsilon^2 \quad \text{and that} \quad M'' = m''. \quad \text{.....(III.)}$$

ART. **69**. We shall give a few examples of the geometrical meaning of the invariants of a linear vector function. (Art. 65 (IV.).)

(1) *The invariant m'' vanishes if the function ϕ transforms a pyramid into another having its edges on the corresponding faces of the old.** If the vectors α, β, γ are along the edges of a pyramid, and if $\phi\alpha$ is coplanar with β and γ, $\phi\beta$ with γ and α, and $\phi\gamma$ with α and β, it is obvious that m'' vanishes. Conversely if m'' vanishes we can determine an infinite number of pyramids which transform into others having their edges on the faces of the originals. For assuming arbitrarily α and β, the equations

$$S\phi\alpha\beta\gamma = 0, \quad S\alpha\phi\beta\gamma = 0, \quad \text{........................(I.)}$$

determine the direction of γ; and the condition $m'' = 0$ requires $S\alpha\beta\phi\gamma = 0$.

(2) *The invariant m' vanishes if ϕ transforms a pyramid into another having its faces through the edges of the old.* The proof and the converse are the same as that just given.

(3) *The sum of the projections of vectors transformed from mutually rectangular unit vectors on the corresponding unit vectors is constant:*

$$m'' = -SU\alpha\phi U\alpha - SU\beta\phi U\rho - SU\gamma\phi U\gamma \quad \text{if} \quad U\beta U\gamma = U\alpha. \text{...(II.)}$$

(4) *The sum of the squares of vectors transformed from mutually rectangular unit vectors is constant:*

$$m''(\phi'\phi) = -\Sigma(\phi U\alpha)^2 = -\Sigma SU\alpha\phi'\phi U\alpha \quad \text{if} \quad U\beta U\gamma = U\alpha \text{ ...(III.)}$$

where $m''(\phi'\phi)$ is the first invariant of the self-conjugate function $\phi'\phi$.

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* In other words if ϕ transforms three planes into planes intersecting in pairs on the original planes.

(5) *The sum of the squares of the projections, on any fixed line, of vectors transformed from mutually rectangular unit vectors is constant.* If λ is the vector on which the others are projected

$$\Sigma(\mathrm{S}\lambda\phi\mathrm{U}\alpha)^2=\Sigma(\mathrm{SU}\alpha\phi'\lambda)^2=\mathrm{T}\phi'\lambda^2. \quad \text{(IV.)}$$

(6) *The sum of the squares of the projections on a plane is constant.*

Similar remarks apply to vector areas $\mathrm{V}\phi\mathrm{U}\alpha\phi\mathrm{U}\beta$, etc.

Ex. 1. If the sum of the square roots of the latent roots of ϕ is zero, it is possible to find an infinite number of pyramids (OABC) which convert into others (OA′B′C′), so that intermediate pyramids ($\mathrm{OA}_{,}\mathrm{B}_{,}\mathrm{C}_{,}$) can be drawn having their three edges in the faces of the first, while their faces contain the edges of the second.

[Here $\mathrm{S}\phi^{\frac{1}{2}}\alpha\beta\gamma=0$, $\mathrm{S}\phi^{\frac{1}{2}}\beta\gamma\alpha=0$, $\mathrm{S}\phi^{\frac{1}{2}}\gamma\alpha\beta=0$, and $\mathrm{S}\phi\alpha\phi^{\frac{1}{2}}\beta\phi^{\frac{1}{2}}\gamma=0$, etc. See the next Article and the Appendix to new edition of *Elements of Quaternions*, vol. ii., note v.]

ART. **70.** The square root of a linear vector function may be defined as a linear vector function, which, operating twice in succession on any vector, produces the same effect as the given function. Writing then $\phi^{\frac{1}{2}}$ for the square root of the function ϕ, we have, if γ_1, γ_2 and γ_3 are the axes of $\phi^{\frac{1}{2}}$, and if h_1, h_2 and h_3 are its roots,

$$\phi^{\frac{1}{2}}\gamma_1=h_1\gamma_1, \quad (\phi^{\frac{1}{2}})^2\gamma_1=h_1{}^2\gamma_1=\phi\gamma_1, \quad \text{(I.)}$$

and consequently the axes of $\phi^{\frac{1}{2}}$ are also axes of ϕ (see Ex. 2, Art. 66), and the squares of the latent roots of $\phi^{\frac{1}{2}}$ are the roots of ϕ. In general, then, a function has eight square roots answering to the double signs attributable to $g_1{}^{\frac{1}{2}}$, $g_2{}^{\frac{1}{2}}$, $g_3{}^{\frac{1}{2}}$. It does not follow conversely, that the axes of ϕ are axes of $\phi^{\frac{1}{2}}$. As an example, let ϕ have equal roots, and let it have indeterminate axes, so that $(\phi-g_1)(x\gamma_1+y\gamma_2)=0$ where x and y are arbitrary, $g_1=g_2$ being the repeated root. A square root of the function may have three distinct roots $+g_1{}^{\frac{1}{2}}$, $-g_1{}^{\frac{1}{2}}$, $g_3{}^{\frac{1}{2}}$. In this case there is an infinite number of square roots, because we may select any vector $x\gamma_1+y\gamma_2$ to be an axis of $\phi^{\frac{1}{2}}$ corresponding to $+g_1{}^{\frac{1}{2}}$, and any other vector $x'\gamma_1+y'\gamma_2$ may be selected as the axis corresponding to $-g_1{}^{\frac{1}{2}}$. For real square roots, the three roots g_1, g_2 and g_3 must of course be positive.

The following resolution of a linear function ϕ and its conjugate is sometimes useful—for example, in the theory of strain. It is due to Tait, to whom is also due the conception of the square root of a linear vector function. 9

Let i, j, k be the mutually rectangular axes of the self-conjugate function $\phi\phi'$, and let a^2, b^2, c^2 be its roots. Reducing ϕ to the trinomial form (Art. 62),

$$\phi\rho=ai\mathrm{S}i'\rho+bj\mathrm{S}j'\rho+ck\mathrm{S}k'\rho, \quad \text{(II.)}$$

where i', j', k' are to be determined, we have $\phi'i=-ai'$, $\phi'j=-bj'$ and $\phi'k=-ck'$. These give $\phi\phi'i=-a^2i\,.\,i'^2-abj\,.\,\mathrm{S}i'j'-ack\,.\,\mathrm{S}i'k'$, but i is by hypothesis an axis of $\phi\phi'$, so that $\phi\phi'i=a^2i$. Consequently we must have $i'^2=-1$, $\mathrm{S}i'j'=\mathrm{S}i'k'=0$, and in fact i', j', k' form a mutually rectangular unit system of vectors. Thus in particular $\phi i'=-ai$, and $\phi'\phi i'=-a\phi i=+a^2i'$, and thus it follows that i', j' and k' are the axes of the *new* self-conjugate function $\phi'\phi$, and that a^2, b^2, c^2 are also its roots.

Let q be a conical rotation which renders* i', j', k' parallel to i, j, k. We have by (II.),

$$\phi\rho = \Sigma ai\mathrm{S}qiq^{-1}\rho = \Sigma ai\mathrm{S}iq^{-1}\rho q\ ;$$

and therefore by the definition of a square root,

$$\phi\rho = (\phi\phi')^{\frac{1}{2}} . q^{-1}\rho q \quad \text{and} \quad \phi'\rho = q . (\phi\phi')^{\frac{1}{2}}\rho . q^{-1}\ ; \quad \text{(III.)}$$

and from these we also deduce

$$q\rho q^{-1} = \phi' . (\phi\phi')^{-\frac{1}{2}}\rho. \quad \text{(IV.)}$$

In like manner we may prove that

$$\phi\rho = p^{-1} . (\phi'\phi)^{\frac{1}{2}}\rho . p, \quad \phi'\rho = (\phi'\phi)^{\frac{1}{2}} . p\rho p^{-1}\ ; \quad \text{(V.)}$$

and thus we can reduce the effect of a function ϕ to a rotation preceded or followed by the operation of a self-conjugate function.

ART. **71.** We add one or two miscellaneous propositions respecting two or more functions.

The functions $\phi\phi_{,}$ and $\phi_{,}\phi$ formed by taking the products of two functions have the same symbolic cubic. For

$$\phi_{,}\phi . \phi_{,}\gamma = g\phi_{,}\gamma \quad \text{if} \quad \phi\phi_{,}\gamma = g\gamma, \quad \text{(I.)}$$

and thus the functions have the same roots and the axes (γ) of $\phi_{,}\phi$ are deducible from those of $\phi\phi_{,}$ by operating with $\phi_{,}$.

In particular $\phi_{,}^{-1}\phi\phi_{,}$ has the same symbolic cubic as ϕ, and thus any peculiarity in the nature of one function occurs also in that of the other.

Any two functions may be reduced simultaneously and generally in one way to the forms

$$\phi\rho = \alpha\mathrm{S}\lambda\rho + \beta\mathrm{S}\mu\rho + \gamma\mathrm{S}\nu\rho\ ; \quad \phi_{,}\rho = a\alpha\mathrm{S}\lambda\rho + b\beta\mathrm{S}\mu\rho + c\gamma\mathrm{S}\nu\rho. \quad \text{(II.)}$$

Assuming the possibility of the reduction, it appears that

$$\phi_{,}\mathrm{V}\mu\nu = a\phi\mathrm{V}\mu\nu = a\alpha\mathrm{S}\lambda\mu\nu, \text{ etc.,}$$

and thus the vectors $\mathrm{V}\lambda\mu$, etc., are the axes of the function $\phi^{-1}\phi_{,}$ and a, b, c are its roots. If both functions are self-conjugate, we must have

$$\mathrm{V}\alpha\lambda + \mathrm{V}\beta\mu + \mathrm{V}\gamma\nu = 0, \quad a\mathrm{V}\alpha\lambda + b\mathrm{V}\beta\mu + c\mathrm{V}\gamma\nu = 0,$$

or

$$\frac{\mathrm{V}\alpha\lambda}{b-c} = \frac{\mathrm{V}\beta\mu}{c-a} = \frac{\mathrm{V}\gamma\nu}{a-b} = 0,$$

and therefore for self-conjugate functions

$$\phi\rho = \lambda\mathrm{S}\lambda\rho + \mu\mathrm{S}\mu\rho + \nu\mathrm{S}\nu\rho, \quad \phi_{,}\rho = a\lambda\mathrm{S}\lambda\rho + b\mu\mathrm{S}\mu\rho + c\nu\mathrm{S}\nu\rho, \quad \text{(III.)}$$

and further it is evident that

$$\mathrm{SV}\mu\nu\phi\mathrm{V}\nu\lambda = 0, \quad \mathrm{SV}\mu\nu\phi_{,}\mathrm{V}\nu\lambda = 0, \text{ etc.}$$

It is sometimes necessary to invert the function $\phi + t\phi_{,}$, and the auxiliary ψ of this function is defined by

$$\psi\mathrm{V}\alpha\beta = \mathrm{V}(\phi' + t\phi_{,}')\alpha(\phi' + t\phi_{,}')\beta = \psi\mathrm{V}\alpha\beta + t\Psi\mathrm{V}\alpha\beta + t^2\psi_{,}\mathrm{V}\alpha\beta \quad \text{(IV.)}$$

where

$$\Psi\mathrm{V}\alpha\beta = \mathrm{V}\phi'\alpha\phi_{,}'\beta + \mathrm{V}\phi_{,}'\alpha\phi'\beta. \quad \text{(V.)}$$

The invariant m_t is

$$m_t = m + lt + l_{,}t^2 + m_{,}t^3 \quad \text{(VI.)}$$

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* We must have $i'j'k' = -1 = ijk$, but this can always be secured by attributing proper signs to a, b, c. If $i'j'k'$ were $+1$, we should not be able to rotate the vectors into ijk, for $qiq^{-1} . qjq^{-1} . qkq^{-1} = q . ijk . q^{-1} = -1$.

where m and $m_{\prime}$ are the third invariants of ϕ and $\phi_{\prime}$ and where l and $l_{\prime}$ are the two new invariants

$$l\mathrm{S}\alpha\beta\gamma=\Sigma\mathrm{S}\phi\alpha\phi\beta\phi_{\prime}\gamma,\quad l_{\prime}\mathrm{S}\alpha\beta\gamma=\Sigma\mathrm{S}\phi\alpha\phi_{\prime}\beta\phi_{\prime}\gamma. \qquad\text{(VII.)}$$

Ex. 1. The locus of axes of the functions $\phi+t\phi_{\prime}$ where t is a scalar parameter is the cubic cone

$$\mathrm{S}\rho\phi\rho\phi_{\prime}\rho=0.$$

[If ρ is an axis $\phi\rho+t\phi_{\prime}\rho=g\rho$. The surface represents a cone, as it is independent of $\mathrm{T}\rho$.]

Ex. 2. The axes of functions of the family $\phi+t\phi_{\prime}$ form co-residual triads on the cubic cone.

[The quadric cone $\mathrm{S}\lambda\rho\phi\rho=0$ in which λ is arbitrary cuts the cubic in the three axes of ϕ and again in three lines in which it cuts $\mathrm{S}\lambda\rho\phi_{\prime}\rho=0$, as we see by substituting $\phi\rho=x\rho+y\lambda$ in the equation of the cubic. The remaining intersection of the quadric cones is $\rho \parallel \lambda$. The cone $\mathrm{S}\lambda\rho(\phi+t\phi_{\prime})\rho=0$ passes through the axes of $\phi+t\phi_{\prime}$ and through the three lines above mentioned, so that these three lines are the residuals of every triad of axes (Salmon's *Higher Plane Curves*, Art. 154). For other properties see *Quaternion Invariants of Linear Vector Functions*, Proc. R.I.A., 1896.]

Ex. 3. Prove that the invariants l and $l_{\prime}$ are merely multiplied by a scalar when ϕ and $\phi_{\prime}$ are replaced by $\phi_1\phi\phi_2$ and $\phi_1\phi_{\prime}\phi_2$.

[The scalar is the product of the third invariants of ϕ_1 and ϕ_2. This very general invariantal property leads to many theorems. See *Phil. Trans.*, vol. 201, Part VIII., sections iii. and x.]

Ex. 4. Prove that the function $\Phi\mathrm{V}\alpha\beta=\mathrm{V}\phi'^{-1}\alpha\phi_{\prime}'^{-1}\beta+\mathrm{V}\phi_{\prime}'^{-1}\alpha\phi'^{-1}\beta$ is co-variant with ϕ and $\phi_{\prime}$.

[Making the substitution of the last example, ϕ'^{-1} becomes $\phi_1'^{-1}\phi'^{-1}\phi_2'^{-1}$ and the function Φ changes into $m_1^{-1}m_2^{-1}\phi_1\Phi\phi_2$.]

Ex. 5. If $\sigma \parallel \mathrm{V}\phi_1\rho\phi_2\rho$ show that $\rho \parallel \mathrm{V}\phi_1'\sigma\phi_2'\sigma$;

and more generally if σ is connected with ρ by the chain of relations

$$\rho_1 \parallel \mathrm{V}\phi_1\rho\phi_2\rho,\quad \rho_2 \parallel \mathrm{V}\phi_3\rho_1\phi_4\rho_1, \ldots\ \sigma \parallel \mathrm{V}\phi_{2n-1}\rho_{n-1}\phi_{2n}\rho_{n-1},$$

prove that an analogous chain of relations connects ρ with σ.

[The second part of this example is related to the theory of the Cremona transformations connecting vectors ρ and σ, the direction of a vector (ρ) being connected by a one-to-one relation with that of a vector (σ).]

Ex. 6. If $\phi(\rho, t)$ is a linear and vector function of ρ and also a function of the scalar t, the equation

$$\mathrm{V}\rho\phi(\rho, t)=0$$

represents a cone whose order is the number of values of t which satisfy

$$\mathrm{S}\lambda\phi'(\lambda, t)\phi'\{\phi'(\lambda, t), t\}=0,$$

λ being any constant vector.

Ex. 7. The equation

$$\mathrm{V}(\rho-\alpha)\phi(\rho, t)=0$$

represents a surface which meets an arbitrary right line $\mathrm{V}(\rho-\beta)\gamma=0$ in as many points as there are values of t which satisfy

$$\mathrm{S}(\beta-\alpha)\gamma\phi(\beta, t)\mathrm{S}\gamma\phi(\beta, t)\phi(\gamma, t)=\mathrm{S}(\beta-\alpha)\gamma\phi(\gamma, t)\mathrm{S}(\beta-\alpha)\phi(\beta, t)\phi(\gamma, t).$$

EXAMPLES TO CHAPTER VIII.

Ex. 1. Find the auxiliary functions χ and ψ and the invariants of the function

$$\phi\rho = \Sigma m \mathrm{V}\alpha\mathrm{V}\rho\alpha.$$

Ex. 2. Invert the function $\phi\rho + \mathrm{V}\alpha\mathrm{V}\rho\alpha$ where ϕ is a given function and where α is a given vector.

Ex. 3. If $\phi_\alpha\rho = \alpha^{-1}\mathrm{V}\alpha\phi\rho$ show that the conjugate function is

$$\phi_\alpha'\rho = \phi'\mathrm{V}\alpha^{-1}\mathrm{V}\alpha\rho,$$

and prove that the spin-vector is $\epsilon - \frac{1}{2}\mathrm{V}\alpha^{-1}\phi'\alpha$.

(*a*) Show that the auxiliary ψ function of $\phi_\alpha + c$ is expressible in either of the forms

$$\psi\alpha\mathrm{S}\alpha^{-1}\rho + c(\chi\rho - \mathrm{V}\phi'\alpha\mathrm{V}\alpha^{-1}\rho) + c^2\rho$$

or

$$(\psi + c\chi + c^2)\rho - \mathrm{V}\phi'\alpha(\phi' + c)\mathrm{V}\alpha^{-1}\rho,$$

and show that the third invariant of the same function is

$$c\mathrm{S}\alpha^{-1}(\psi + c\chi + c^2)\alpha.$$

(*b*) Prove that the axes of ϕ_α are determined by substituting a root of the equation $c\mathrm{S}\alpha(\psi + c\chi + c^2)\alpha = 0$ in $(\phi + c)^{-1}\alpha$.

Ex. 4. If $\phi_{\prime}\rho = \phi\rho + \alpha\mathrm{S}\beta\rho$, show that $\psi_{\prime}\rho = \psi\rho + \mathrm{V}\beta\phi'\mathrm{V}\alpha\rho$ and that $m_{\prime} = m + \mathrm{S}\beta\psi\alpha$.

Ex. 5. Show that the ψ function and the third invariant of $\phi\rho - \mathrm{V}\beta\mathrm{V}\alpha\rho$ may be reduced to the forms

$$\psi\rho - \chi\alpha\mathrm{S}\beta\rho - \mathrm{V}\phi'\beta\mathrm{V}\alpha\rho + \alpha\mathrm{S}\beta\rho\mathrm{S}\alpha\beta$$

and

$$m - \mathrm{S}\beta\phi\chi\alpha + \mathrm{S}\beta\phi\alpha\mathrm{S}\alpha\beta.$$

Ex. 6. If $\phi_c = \phi + c$, etc., show that

$$\chi_c = \chi + 2c, \quad m_c' = m' + 2m''c + 3c^2, \quad m_c'' = m'' + 3c.$$

Ex. 7. Prove that

$$\mathrm{V}\,.\,\phi\mathrm{V}\alpha\rho\,.\,\beta = \chi'\mathrm{V}\,.\,\mathrm{V}\alpha\rho\,.\,\beta - \mathrm{V}\,.\,\mathrm{V}\alpha\rho\,.\,\phi\beta.$$

(*a*) Show that the conjugate of this linear function of ρ is $\mathrm{V}\,.\,\phi'\mathrm{V}\beta\rho\,.\,\alpha$, and prove that the spin-vector is $\frac{1}{2}\phi'\mathrm{V}\alpha\beta - \alpha\mathrm{S}\epsilon\beta$ where ϵ is the spin-vector of ϕ.

(*b*) Show that the auxiliary ψ function is $\alpha\mathrm{S}\beta\rho\mathrm{S}\alpha\psi\beta$.

(*c*) If $\mathrm{V}\,.\,\phi\mathrm{V}\alpha\rho\,.\,\beta = \sigma$, show that $\rho = x\alpha - \phi'\sigma(\mathrm{S}\alpha\psi\beta)^{-1}$ where x is an arbitrary scalar. Deduce this result by the aid of the implied relations $\mathrm{S}\alpha\rho\phi'\sigma = 0$, $\mathrm{S}\beta\sigma = 0$.

Ex. 8. Prove that

$$\nabla\,.\,\phi(\rho) = -\Sigma\mathrm{V}\beta\gamma\,.\,\phi\alpha(\mathrm{S}\alpha\beta\gamma)^{-1}$$

where α, β and γ are arbitrary vectors.

(*a*) Show that

$$\nabla\,.\,\psi(\rho) = -\Sigma\alpha\,.\,\mathrm{V}\phi'\beta\phi'\gamma\,.\,(\mathrm{S}\alpha\beta\gamma)^{-1}.$$

(*b*) Express these quaternions in terms of the scalar invariants and the spin-vectors.

Ex. 9. Three lines are defined by the pairs of vectors (σ_1, τ_1), (σ_2, τ_2), (σ_3, τ_3) as in Art. 36, Ex. 4, show that any line which is met by all the transversals of the given lines may be represented by

$$\sigma = \phi\tau \quad \text{where} \quad S\tau\phi\tau = 0,$$

the linear function ϕ being defined by the equations

$$\sigma_1 = \phi\tau_1, \quad \sigma_2 = \phi\tau_2, \quad \sigma_3 = \phi\tau_3.$$

(*a*) The transversals of the same set of lines may be represented by

$$\sigma' = -\phi'\tau' \quad \text{where} \quad S\tau'\phi'\tau' = 0,$$

the function ϕ' being the conjugate of ϕ.

(*b*) Writing

$$\sigma = \phi\tau = V\rho\tau$$

and expressing that the function $\phi(\ ) - V\rho(\ )$ has a zero root, the locus of the lines is found to be

$$S(\rho - \epsilon)\phi(\rho - \epsilon) = m + S\epsilon\phi\epsilon$$

where m is the third invariant of the function ϕ and where ϵ is its spin-vector.

(*c*) The same equation is satisfied by the transversals.

(*d*) Show that four given lines have in general two common transversals; and that these are determined by

$$\sigma' = -\phi'\tau' \quad \text{where} \quad S\tau'(\sigma_4 - \phi\tau_4) = 0, \quad S\tau'\phi'\tau' = 0,$$

the fourth line being defined by (σ_4, τ_4).

Ex. 10. Given any four pairs of vectors, (β_n, α_n), where $n = 1, 2, 3$ or 4, show how to find a linear vector function ϕ and a vector γ so that

$$\beta_n = \phi\alpha_n + \gamma.$$

Ex. 11. Given any six triads of vectors $(\gamma_n, \beta_n, \alpha_n)$ where $n = 1, 2, \ldots 6$; determine two linear functions ϕ_1 and ϕ_2 so that

$$\gamma_n = \phi_1\alpha_n + \phi_2\beta_n.$$

Ex. 12. Verify by assuming $\rho = x\alpha + y\beta$, $S\lambda\alpha = 0$, $S\lambda\beta = 0$, that the solutions of the equations $S\lambda\rho = 0$, $S\rho\phi\rho = 0$ may be written in the form

$$\rho = V\phi\alpha\lambda \pm \alpha(S\lambda\psi\lambda)^{\frac{1}{2}}$$

where α is any vector perpendicular to λ.

Ex. 13. Given two tetrahedra A′B′C′D′ and ABCD, find a point E and a function ϕ so that

$$EA' = \phi . EA, \quad EB' = \phi . EB, \quad EC' = \phi . EC, \quad ED' = \phi . ED.$$

(*a*) Show that corresponding faces of the tetrahedron determine with the point E tetrahedra having a common ratio of volumes.

(*b*) If the lines joining corresponding vertices are generators of the same system of a hyperboloid, it is possible to find four scalars l, m, n, p so that

$$l(\alpha' - \alpha) + m(\beta' - \beta) + n(\gamma' - \gamma) + p(\delta' - \delta) = 0;$$
$$lV\alpha\alpha' + mV\beta\beta' + nV\gamma\gamma' + p\delta\delta' = 0.$$

(*c*) These scalars are independent of the origin, and if the origin is taken at the point E, we shall have

$$l\alpha+m\beta+n\gamma+p\delta=0,\quad l\alpha'+m\beta'+n\gamma'+p\delta'=0$$

for an arbitrary pair of tetrahedra, while if the lines joining the vertices are generators of the same system of a hyperboloid, we shall have in addition

$$l\mathrm{V}\alpha\alpha'+m\mathrm{V}\beta\beta'+n\mathrm{V}\gamma\gamma'+p\mathrm{V}\delta\delta'=0.$$

Ex. 14. Identify the expressions

$$\rho=(\phi+t)^{-1}(\lambda+t\mu)=\frac{(\alpha,\ \beta,\ \gamma,\ \delta \between t,\ 1)^3}{(a,\ b,\ c,\ d \between t,\ 1)^3}$$

where t is a scalar variable, and show how to express the function ϕ, and the vectors λ and μ in terms of the vectors α, β, γ and δ, and the scalars a, b, c and d.

Ex. 15. Of what nature are the curve loci

$$\rho=(\phi+t)^{-1}(\alpha+t\beta)\ \text{ and }\ \rho=(\phi+t)(\alpha+t\beta)^{-1}\,?$$

Ex. 16. Gauss has described, in an unpublished MS. of the year 1819, an operator which alters the size of any figure in a given ratio, and which turns the figure through a given angle round a given line through the origin. He proves that an operator of this kind depends on four numbers, that successive operators compound into a single operator of the same kind, and that the order of the operations is not commutative.

(*a*) Show that Gauss's operator may be expressed in quaternions by $cq(\ \ )q^{-1}$, c being a given scalar, and q a given quaternion.

(*b*) Hence prove his theorems.

(*c*) Compare and contrast the lack of commutation in the order of these operators, or in the order of the operators 2 and cos. in the simple inequality

$$\cos 2x \gtrless 2\cos x,$$

with the lack of commutation in the multiplication of quaternions.

(*d*) Prove that the sum of two Gaussian operators is an operator of a distinct kind.

(*e*) Prove that a sum of at least three Gaussian operators is required to adequately express a linear vector function. (Bishop Law's Premium, 1899.)

Ex. 17. Unit vectors α, β and γ are directed respectively from the centre of a regular solid to the middle point of a face (or to a vertex); to the middle point of an edge of the face (or of an edge through the vertex); and to a vertex on that edge (or to the middle point of a face containing the edge), prove that

$$\gamma^{\frac{4}{3}}=\beta\alpha^{\frac{2}{n}},$$

where $n=3$ for the tetrahedron, $n=4$ for the cube and octahedron, and $n=5$ for the dodecahedron and ikosahedron.

(*a*) Hence show that all rotations which leave unchanged the region occupied by the solid may be represented by powers and products of linear vector functions λ, κ and ι which obey the laws *

$$\lambda^n=1,\quad \kappa^3=1,\quad \iota^2=1,\quad \lambda=\iota\kappa,\quad (n=3,\ 4 \text{ or } 5).$$

* See Hamilton on the *Icosian Calculus*, Phil. Mag., Dec. 1856; Proc. R.I.A., Vol. VI., pp. 415, 416. See also Burnside's *Theory of Groups*, Arts. 200, *et seq.*

Ex. 18. A real linear function which is a symbolical n^{th} root of unity, or which satisfies the equation

$$\phi^n = 1$$

is of the form

$$\phi\rho \,.\, S\alpha\beta\gamma = \left(\alpha\cos\frac{2\pi}{n} - \beta\sin\frac{2\pi}{n}\right) S\beta\gamma\rho + \left(\alpha\sin\frac{2\pi}{n} + \beta\cos\frac{2\pi}{n}\right) S\gamma\alpha\rho + \gamma S\alpha\beta\rho,$$

where α, β and γ are arbitrary real vectors.

Ex. 19. The result of eliminating the vector ϖ between the equations

$$S\varpi\alpha = 0, \quad S\varpi\phi\rho = 0, \quad S\varpi\phi\varpi = 0$$

may, when ϕ is self-conjugate, be expressed in the form

$$S\alpha\psi\alpha S\rho\phi\rho - mS\alpha\rho^2 = 0.$$

(a) In the same case,

$$S\phi\rho\alpha\nabla \,.\, V\alpha\phi\rho = V\alpha\phi V\alpha\phi\rho = \rho S\alpha\psi\alpha - \psi\alpha S\alpha\rho.$$

(b) And moreover *

$$S\phi\rho\alpha\lambda S\phi\rho\alpha\mu = S\rho\phi\rho SV\alpha\lambda\phi V\alpha\mu + S\lambda\psi\mu S\alpha\rho^2 - S\alpha\psi\mu S\lambda\rho S\alpha\rho - S\alpha\psi\lambda S\mu\rho S\alpha\rho + S\alpha\psi\alpha S\lambda\rho S\mu\rho.$$

* These examples are quaternion equivalents of the transformations in Arts. 383, 385 and 390 of Salmon's *Higher Plane Curves.*

CHAPTER IX.

QUADRIC SURFACES.

ART. **72.** If $f(\rho, \rho)$ is a homogeneous, rational and integral scalar function of the second order in a variable vector ρ, so that

$$f(\alpha+t\beta, \alpha+t\beta)=f(\alpha, \alpha)+t(f(\alpha, \beta)+f(\beta, \alpha))+t^2 f(\beta, \beta), \quad \ldots(\text{I.})$$

where α and β are arbitrary vectors, the equation

$$f(\rho, \rho)=\text{const.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{II.})$$

represents a surface of the second order, referred to its centre as origin. For by (I.) we find a quadratic in t which determines two points in which an arbitrary line $\rho=\alpha+t\beta$ cuts the surface; and on putting $\alpha=0$, the roots of the quadratic are equal and opposite, showing that every chord through the origin is bisected at that point.

The coefficient of t in (I.) is linear and homogeneous both in α and in β, and as it involves these vectors symmetrically we may write

$$f(\alpha, \beta)+f(\beta, \alpha)=2S\alpha\phi\beta=2S\beta\phi\alpha \quad \ldots\ldots\ldots\ldots(\text{III.})$$

where ϕ is a self-conjugate linear vector function. Thus the equation of the central quadric is expressible in the form

$$f(\rho, \rho)=S\rho\phi\rho=\text{const.} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{IV.})$$

Without loss of generality we may suppose the constant incorporated in ϕ, and we take as the equation

$$S\rho\phi\rho=-1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{V.})$$

in which, as we have said, ϕ is self-conjugate. Of course, and without gain of generality, we may suppose ϕ not to be self-conjugate in (V.), for the spin-vector automatically disappears from an equation of this form (Art. 67); but this is very likely to lead to mistakes in further developments, and it adds needless complexity.

ART. **73.** Equation (v.) of the last article gives

$$\mathrm{T}\rho^2\mathrm{SU}\rho\phi\mathrm{U}\rho = -1 \quad \text{or} \quad -\mathrm{SU}\rho\phi\mathrm{U}\rho = \frac{1}{\mathrm{T}\rho^2} = \frac{1}{r^2} \qquad \text{(I.)}$$

if r is the length of the central radius parallel to $\mathrm{U}\rho$.

For a closed quadric, an ellipsoid or sphere, r^2 is always positive, as every line through the centre meets the closed surface in real points. For a hyperboloid, the radius becomes infinite for an edge of the cone

$$\mathrm{SU}\rho\phi\mathrm{U}\rho = 0 \quad \text{or} \quad \mathrm{S}\rho\phi\rho = 0, \qquad \text{(II.)}$$

the asymptotic cone of the surface. The sign of the expression r^{-2} or $-\mathrm{SU}\rho\phi\mathrm{U}\rho$ changes on passing through a zero value, and the expression remains with changed sign until it passes again through a zero value. So on one side of the cone $\mathrm{S}\rho\phi\rho = 0$, lines meet the hyperboloid in real points, and on the other side the points are imaginary and the corresponding vectors are of the form $\rho = \sqrt{-1}\rho'$, $(\mathrm{U}\rho = \mathrm{U}\rho',\ \mathrm{T}\rho = \sqrt{-1}\mathrm{T}\rho')$, where ρ' is a real vector.

The vectors ρ' terminate on the quadric

$$\mathrm{S}\rho\phi\rho = +1 \qquad \text{(III.)}$$

—the conjugate of the quadric $\mathrm{S}\rho\phi\rho = -1$.

For the sake of brevity we shall write generally r^2 for the square of the length of the radius whether that square be positive or negative, the interpretation in the latter case being that just given.

An arbitrary right line $\rho = \alpha + t\beta$ cuts the quadric $\mathrm{S}\rho\phi\rho = -1$ in the points determined by the roots t of the quadratic

$$\mathrm{S}\alpha\phi\alpha + 2t\mathrm{S}\alpha\phi\beta + t^2\mathrm{S}\beta\phi\beta = -1. \qquad \text{(IV.)}$$

For a real and positive root, the point is in the direction $+\mathrm{U}\beta$ from the extremity of α, and for a negative root it is in the direction $-\mathrm{U}\beta$. For equal roots, the line touches the surface; and for imaginary it cuts it in imaginary points.

The locus of the middle points of chords parallel to β is the diametral plane

$$\mathrm{S}\rho\phi\beta = 0, \qquad \text{(V.)}$$

for if α is the vector to any point in this plane, the roots of (IV.) are equal and opposite. If the diametral plane of β contains the vector α, that of α contains β in virtue of the self-conjugate property of ϕ, for then

$$\mathrm{S}\alpha\phi\beta = \mathrm{S}\beta\phi\alpha = 0. \qquad \text{(VI.)}$$

The equation has equal roots if

$$\mathrm{S}\beta\phi\beta(\mathrm{S}\alpha\phi\alpha + 1) - (\mathrm{S}\alpha\phi\beta)^2 = 0, \qquad \text{(VII.)}$$

and regarding β as variable, this is the equation of the tangent cone from the extremity of the vector α referred to that extremity as origin, for it is independent of $T\beta$. Replacing β by $\rho-\alpha$, the equation of the same cone referred to the centre as origin easily reduces to

$$(S\rho\phi\rho+1)(S\alpha\phi\alpha+1)-(S\rho\phi\alpha+1)^2=0\,; \quad \text{.........(VIII.)}$$

and the form of the equation shows that the cone touches the quadric along its intersection with the plane

$$S\rho\phi\alpha=-1 \quad \text{...............................(IX.)}$$

—the polar plane of the extremity of α.

If the vector α terminates on the surface, the equation of the cone becomes the square of the equation of a plane—the tangent plane at the extremity of α,

$$S\rho\phi\alpha=-1, \quad S\alpha\phi\alpha=-1. \quad \text{.......................(X.)}$$

Allowing on the other hand α to vary arbitrarily in the quadratic equation, and putting for greater clearness $\alpha=\rho'=\rho-t\beta$, the vector ρ' being drawn from the extremity of the vector $t\beta$ while ρ is drawn from the centre, we see that

$$S\rho'\phi\rho'=-1-t^2S\beta\phi\beta \quad \text{if} \quad S\rho'\phi\beta=0. \quad \text{.............(XI.)}$$

These two equations jointly represent the section of the quadric by the plane

$$S\rho\phi\beta=tS\beta\phi\beta, \quad \text{..........................(XII.)}$$

and the centre of the section is the origin of vectors ρ', or the extremity of the vector $t\beta$. Hence the locus of centres of sections by planes parallel to (V.) is the line through the centre parallel to β, as indeed might have been proved directly from (V.). The section (XI.) is similar to the parallel central section of the quadric, for if r' is the radius of the section parallel to ρ' and r that of the quadric,

$$-r'^2SU\rho'\phi U\rho'=\frac{r'^2}{r^2}=1+t^2S\beta\phi\beta=1-\frac{T(t\beta)^2}{b'^2} \quad \text{.....(XIII.)}$$

if b' is the radius of the quadric parallel to β.

The equation of the normal to the quadric at the extremity of the vector α is

$$\rho=\alpha+x\phi\alpha, \quad \text{or} \quad V(\rho-\alpha)\phi\alpha=0\,; \quad \text{...........(XIV.)}$$

and the normals which pass through a given point β are six in number and are determined by the equation

$$\beta=\rho+x\phi\rho, \quad \text{or} \quad V(\beta-\rho)\phi\rho=0, \quad \text{and} \quad S\rho\phi\rho=-1. \quad \text{...(XV.)}$$

To solve these equations we have

$$\rho=(1+x\phi)^{-1}\beta, \quad \text{where} \quad S\beta\phi(1+x\phi)^{-2}\beta=-1, \quad \text{.....(XVI.)}$$

because $S\rho\phi\rho=-1$, and on inversion we find a sextic equation n x.

Ex. 1. Prove that the rectangle under the intercepts from the extremity of α on the line $\rho=\alpha+t\beta$ is

$$(\mathrm{T}\alpha^2-a'^2)b'^2a'^{-2}$$

where a' and b' are the central radii parallel to α and β.

[$t_1t_2\mathrm{T}\beta^2=(\mathrm{S}\alpha\phi\alpha+1):\mathrm{SU}\beta\phi\mathrm{U}\beta$.]

Ex. 2. The ratio of the rectangles under the intercepts of lines drawn from a fixed point is independent of the position of the point, and is equal to the ratio of the squares of parallel central radii.

Ex. 3. Chords drawn through a point are divided harmonically by the quadric and the polar plane of the point.

[Put $\dfrac{2}{\rho-\alpha}=\dfrac{t_1}{\beta}+\dfrac{t_2}{\beta}$ where t_1 and t_2 are the roots of the quadratic (IV.).]

Ex. 4. Find the central vector perpendicular on the tangent plane at any point, and obtain the locus of the feet of central perpendiculars, or the central pedal surface.

[$\varpi=-(\phi\alpha)^{-1}$; $\alpha=-\phi^{-1}\varpi^{-1}$; $\mathrm{S}\varpi^{-1}\phi^{-1}\varpi^{-1}=-1$.]

Ex. 5. Prove that the central pedal surface is the inverse of the reciprocal quadric.

Ex. 6. Prove that the ratio of the perpendiculars from a point A and from the centre on the polar plane of B is equal to the ratio of the perpendiculars from B, and from the centre on the polar plane of A.

Ex. 7. Find the locus of the poles of tangent planes to the surface $\mathrm{S}\rho\phi_1\rho=-1$ with respect to the surface $\mathrm{S}\rho\phi_2\rho=-1$.

Ex. 8. Find the pedal surface for an arbitrary point.

Ex. 9. The feet of the normals which pass through a given point are the intersections of a twisted cubic with the quadric.

[Compare (XV.) and Art. 65, Ex. 8, p. 93.]

Ex. 10. The normals through a given point lie on a quadric cone $\mathrm{S}(\rho-\beta)\phi\beta\phi\rho=0$, and the feet of the normals lie on the cone $\mathrm{S}\beta\rho\phi\rho=0$.

(*a*) Both these cones have edges parallel to the three axes.

Ex. 11. Find the condition of the intersection of normals at two points α and β.

Ex. 12. Find the equation of the polar plane of α to the quadric $\mathrm{S}\rho\phi_1\phi_2\rho=-1$, $\phi_1\phi_2$ being the product of two linear functions.

[Note that $\phi_2'\phi_1'$ is the conjugate of $\phi_1\phi_2$.]

Ex. 13. Prove that the polar line of $\rho=\alpha+t\beta$ with respect to the quadric $\mathrm{S}\rho\phi\rho=-1$ is

$$\rho=\frac{\phi\beta+s}{\psi\mathrm{V}\beta\alpha}.$$

ART. **74.** The central plane $\mathrm{S}\lambda\rho=0$ is the diametral plane of chords parallel to $\phi^{-1}\lambda$, as appears on comparison with (V.) of the last article. The locus of the centres of sections by planes parallel to $\mathrm{S}\lambda\rho=0$ is the right line

$$\mathrm{V}\rho\phi^{-1}\lambda=0. \qquad \text{(I.)}$$

The vector to the pole of the plane (Art. 73 (IX.))

$$S\lambda\rho = -1 \text{ is } \phi^{-1}\lambda; \quad \text{.........................(II.)}$$

and the plane touches the quadric if (Art. 73 (X.))

$$S\lambda\phi^{-1}\lambda = -1, \quad \text{.........................(III.)}$$

and as λ varies this is the tangential equation of the quadric. But $S\lambda\rho = -1$ is the polar plane of the extremity of λ with respect to the unit sphere, $T\rho = 1$ or $\rho^2 = -1$, and the equation (III.) may therefore be regarded as that of the reciprocal of the quadric with respect to the unit sphere.

The vector to the centre of the section by $S\lambda\rho = -1$ is by (I.)

$$-\frac{\phi^{-1}\lambda}{S\lambda\phi^{-1}\lambda}, \quad \text{.........................(IV.)}$$

the tensor being determined so that this vector may terminate in the plane $S\lambda\rho = -1$; and on comparison with (XIII.) of the last article, the ratio of the radii is given by

$$\frac{r'^2}{r^2} = \frac{1 + S\lambda\phi^{-1}\lambda}{S\lambda\phi^{-1}\lambda}. \quad \text{.........................(V.)}$$

Ex. 1. By direct comparison of $S\lambda\rho + 1 = 0$ with (XII.) of the last article, find the vector (IV.) of the present.

Ex. 2. Find the reciprocal of the surface with respect to an arbitrary sphere.

Ex. 3. Find the lines in which the plane $S\lambda\rho = 0$ cuts the cone $S\rho\phi\rho = 0$; and show that they are parallel to

$$V\lambda\phi\alpha \pm \alpha(S\lambda\psi\lambda)^{\frac{1}{2}}$$

where α is an arbitrary vector in the plane.

[Assume the lines to be $\alpha + t\alpha'$ where $V\alpha\alpha' = \lambda$ and actually solve for t on substitution in the equation of the cone.]

Ex. 4. Prove that the tangent of the angle between the lines in which the plane $S\lambda\rho = 0$ cuts the cone $S\rho\phi\rho = 0$ is

$$\tan u = 2\frac{T\lambda(S\lambda\psi\lambda)^{\frac{1}{2}}}{S\lambda\chi\lambda}.$$

[If $\alpha + t_1\alpha'$ and $\alpha + t_2\alpha'$ are the lines, calculate $\alpha^2 + (t_1 + t_2)S\alpha\alpha' + t_1t_2\alpha'^2$ and $(t_1 - t_2)V\alpha\alpha'$.]

Ex. 5. Show that the lines in which the plane $S\lambda\rho = 0$ cuts the cone $S\rho\phi\rho = 0$ are parallel to the vectors

$$V\lambda[V\psi\lambda\lambda \pm \phi\lambda(S\lambda\psi\lambda)^{\frac{1}{2}}].$$

ART. **75.** The vector radii α and β of the quadric are conjugate if

$$S\alpha\phi\beta = 0, \quad \text{.........................(I.)}$$

that is if one lies in the diametral plane of the other (Art. 73 (VI.)); and it follows geometrically, or directly from the equations of the tangent planes

$$S\rho\phi\alpha = -1,\ S\rho\phi\beta = -1;\ S\alpha\phi\alpha = -1,\ S\beta\phi\beta = -1, \quad \text{.....(II.)}$$

at the extremities of these vectors, that each vector is parallel to the tangent plane corresponding to the other.

If the vectors are perpendicular as well as conjugate, they are the principal axes of the section by their plane, and the conditions are

$$S\alpha\phi\beta = S\alpha\beta = 0. \quad \text{(III.)}$$

From these we see that

$$\beta \parallel V\alpha\phi\alpha, \quad \alpha \parallel V\beta\phi\beta; \quad \text{(IV.)}$$

so that if one vector is given, the other is determinate; or given that a line is to be the principal axis of a section, the other principal axis is determined by (IV.), and the normal to the section is parallel to

$$V\alpha\beta \parallel \alpha V\alpha\phi\alpha \parallel \phi\alpha \cdot \alpha^2 - \alpha S\alpha\phi\alpha \parallel \phi\alpha T\alpha^2 - \alpha. \quad \text{(V.)}$$

Thus to determine the principal axes in a central plane $S\lambda\rho = 0$, we have

$$\phi\alpha T\alpha^2 - \alpha \parallel \lambda \quad \text{or} \quad \alpha \parallel (\phi T\alpha^2 - 1)^{-1}\lambda; \quad \text{(VI.)}$$

and because $S\lambda\alpha = 0$, we have if $T\alpha^2 = r^2$,

$$S\lambda(\phi r^2 - 1)^{-1}\lambda = 0 \quad \text{or} \quad r^4 S\lambda\psi\lambda - r^2 S\lambda\chi\lambda + \lambda^2 = 0, \quad \text{(VII.)}$$

using the formula of inversion (Art. 65). Thus a quadratic in r^2 is obtained and substitution of its roots in $(\phi r^2 - 1)^{-1}\lambda$ gives the directions of the vectors required.

The principal axes of a surface are normal to the tangent planes at their extremities, so that

$$V\rho\phi\rho = 0 \quad \text{(VIII.)}$$

for a principal axis. These are the axes γ_1, γ_2, γ_3 of the function ϕ.

Ex. 1. Find the maximum and minimum radii in a central section.

[Here $S\lambda\rho = 0$, $S\rho\phi\rho = -1$, $T\rho = \max.$, and on differentiation, $S\lambda d\rho = 0$, $S\phi\rho d\rho = 0$, $S\rho d\rho = 0$, so that the three vectors λ, $\phi\rho$ and ρ are coplanar, or $(\phi + x)\rho = y\lambda$. Operating by $S\rho$, we fall back on (VI.).

Ex. 2. Find the maximum and minimum radii of the quadric, and show that their directions are the solutions of

$$V\rho\phi\rho = 0.$$

Ex. 3. The sections by planes perpendicular to λ are rectangular hyperbolas if

$$S\lambda\chi\lambda = 0.$$

Ex. 4. The equations (IV.) fail in one case.

[Where the vector α is a principal axis of the surface.]

Ex. 5. In general, the three radii are coplanar which are axes of sections having any three mutually rectangular radii as the remaining axes.

[Because ϕ is self-conjugate, $V\alpha^{-1}\phi\alpha + V\beta^{-1}\phi\beta + V\gamma^{-1}\phi\gamma = 0$ if α, β and γ are mutually perpendicular (Art. 67, Ex. 8, p. 97).]

Ex. 6. The sum of the squares of the reciprocals of three mutually rectangular radii is constant.

Ex. 7. Interpret geometrically the equation

$$S\lambda\left(\phi-\frac{1}{r^2}\right)^{-1}\lambda=0,$$

which asserts that the plane $S\lambda\rho=0$ cuts the quadric in a section having a principal axis equal to r.

[This expresses that the plane touches a certain cone.]

Ex. 8. Central planes cut a quadric in sections of given area A. Prove that their envelope is the cone

$$S\rho\left(\psi-\frac{\pi^2}{A^2}\right)^{-1}\rho=0.$$

Ex. 9. The axes of the section by the plane $S\lambda\rho+1=0$ are the roots of the quadratic

$$S\lambda\left(\phi-\frac{m+S\lambda\psi\lambda}{r^2 S\lambda\psi\lambda}\right)^{-1}\lambda=0.$$

Ex. 10. The area of the section made by the plane $S\lambda\rho+1=0$ is

$$A=\pi\frac{T\lambda(m+S\lambda\psi\lambda)}{(-S\lambda\psi\lambda)^{\frac{3}{2}}}.$$

ART. **76.** From any pair of conjugate radii α and β we can derive a third radius conjugate to both so that

$$S\beta\phi\gamma=S\gamma\phi\alpha=S\alpha\phi\beta=0. \quad\text{(I.)}$$

We may in fact regard the two conditions in γ as equations of planes, and

$$\gamma \parallel V\phi\alpha\phi\beta \parallel \psi V\alpha\beta \parallel \phi^{-1}V\alpha\beta. \quad\text{(II.)}$$

With proper tensor the radius γ is

$$\gamma=\frac{\phi^{-1}V\alpha\beta}{\sqrt{(-SV\alpha\beta\phi^{-1}V\alpha\beta)}}. \quad\text{(III.)}$$

In terms of the three mutually conjugate radii, the equation of the quadric is

$$(S\beta\gamma\rho)^2+(S\gamma\alpha\rho)^2+(S\alpha\beta\rho)^2=(S\alpha\beta\gamma)^2 \quad\text{(IV.)}$$

as appears on substituting $\rho=\Sigma\alpha S\beta\gamma\rho : S\alpha\beta\gamma$ in $S\rho\phi\rho=-1$ and attending to the conditions.

Writing (compare Art. 70)

$$\alpha=\phi^{-\frac{1}{2}}\alpha',\quad \beta=\phi^{-\frac{1}{2}}\beta',\quad \gamma=\phi^{-\frac{1}{2}}\gamma' \quad\text{(V.)}$$

it appears by (I.) that the vectors α', β' and γ' are mutually perpendicular, and because α, β and γ terminate upon the surface $S\rho\phi\rho=-1$, it further appears that α', β' and γ' are unit vectors. The theorems of Art. 69 therefore apply, the vectors α, β and γ being the results of operating by a linear function ($\phi^{-\frac{1}{2}}$) on three mutually rectangular unit vectors. Thus the sum of the squares of three mutually conjugate radii is constant, etc.

Ex. 1. The radii α, β, γ being mutually conjugate, prove that

$$\phi\alpha=-\frac{V\beta\gamma}{S\alpha\beta\gamma},\quad \phi\beta=-\frac{V\gamma\alpha}{S\alpha\beta\gamma},\quad \phi\gamma=-\frac{V\alpha\beta}{S\alpha\beta\gamma};$$

and that

$$\frac{m'}{m}=T\alpha^2+T\beta^2+T\gamma^2;\quad \frac{m''}{m}=TV\beta\gamma^2+TV\gamma\alpha^2+TV\alpha\beta^2;\quad m=(S\alpha\beta\gamma)^{-2}.$$

Ex. 2. The locus of the extremity of the diagonal of a parallelepiped having three mutually conjugate radii as conterminous sides is

$$S\rho\phi\rho+3=0.$$

Ex. 3. The locus of the mean point of a triangle formed by the extremities of mutually conjugate radii is

$$S\rho\phi\rho+\frac{1}{3}=0.$$

Ex. 4. The locus of a point from which it is possible to draw three tangents parallel to mutually conjugate radii is

$$S\rho\phi\rho+\frac{3}{2}=0.$$

Ex. 5. In the last example show that a point on the locus is

$$\rho=\frac{1}{\sqrt{2}}(\alpha+\beta+\gamma);$$

and that the points of contact are the extremities of

$$\frac{1}{\sqrt{2}}(\beta+\gamma),\quad \frac{1}{\sqrt{2}}(\gamma+\alpha),\quad \frac{1}{\sqrt{2}}(\alpha+\beta).$$

ART. **77.** To find the cyclic planes of a quadric we have to throw its equation into the form

$$S\rho\phi\rho=g\rho^2+2S\lambda\rho S\mu\rho=-1 \quad\text{......................(I.)}$$

or to determine g, λ and μ so that for *all* vectors ρ,

$$\phi\rho=g\rho+\lambda S\mu\rho+\mu S\lambda\rho. \quad\text{........................(II.)}$$

It follows that $\phi-g$ must reduce every vector to a fixed plane, that of λ and μ. The scalar g must therefore be one of the latent roots of ϕ, say $g=g_2$, and in terms of the axes,

$$S\rho(\phi-g_2)\rho=-(g_1-g_2)(Si\rho)^2-(g_3-g_2)(Sk\rho)^2=2S\lambda\rho S\mu\rho \quad\text{(III.)}$$

because $\qquad \phi\rho=-ig_1Si\rho-jg_2Sj\rho-kg_3Sk\rho.$

Thus

$$t\lambda=\sqrt{g_2-g_1}\,i+\sqrt{g_3-g_2}\,k,\quad 2t^{-1}\mu=\sqrt{g_2-g_1}\,i-\sqrt{g_3-g_2}\,k \quad\text{(IV.)}$$

where t is arbitrary. The transformation is real only if

$$g_1 > g_2 > g_3 \quad \text{or} \quad g_3 > g_2 > g_1. \qquad \text{(v.)}$$

The cyclic planes $S\lambda\rho = 0$, $S\mu\rho = 0$ cut the surface in circles of radius $g_2^{-\frac{1}{2}}$, and these circles are real only if $g_2 > 0$.

The planes $S\lambda\rho + l = 0$, $S\mu\rho + m = 0$ cut the surface in circles lying on the sphere

$$S\rho\phi\rho + 1 - 2(S\lambda\rho + l)(S\mu\rho + m) = 0,$$

or

$$g\rho^2 - 2S\rho(l\mu + m\lambda) - 2lm + 1 = 0.$$

In nearly every problem relating to quadrics some valuable information will be gained by throwing the equation into the cyclic form or into the focal form of the next article. This transformation is not generally of any great difficulty.

Ex. 1. Reduce a quadric to the form

$$T(\rho - \alpha)^2 = e(S\lambda\rho + 1)(S\mu\rho + 1).$$

[This gives Dr. Salmon's focal property. The locus of the extremity of the vector α is a hyperbola—the focal hyperbola, and this depends on equation (IV.).]

Ex. 2. Prove that the roots for Hamilton's cyclic form are

$$g, \quad g + S\lambda\mu + T\lambda\mu, \quad g + S\lambda\mu - T\lambda\mu.$$

Ex. 3. Any two circular sections of opposite systems lie on the same sphere.

Ex. 4. If a quadric is a surface of revolution,

$$V\rho\chi\rho V\chi\rho\psi\rho = (V\rho\psi\rho)^2$$

for all vectors ρ.

[The self-conjugate function ϕ has two equal roots (c) and (Art. 66 (XII.), p. 95)

$$V(\phi - c)\alpha(\phi - c)\beta$$

is identically zero for all vectors α and β, or $\psi\rho - c\chi\rho + c^2\rho = 0$.]

Ex. 5. If for all vectors ρ

$$S\rho\chi\rho\psi\rho = 0, \quad \text{or} \quad S\rho\phi\rho\phi^2\rho = 0, \quad \text{or} \quad S\rho\phi\rho\psi\rho = 0,$$

the quadric is of revolution.

Ex. 6. From a fixed point A, on the surface of a given sphere, draw any chord AD; let D′ be the second point of intersection of the same spheric surface with the secant BD drawn from a fixed external point B; and take a radius vector AE, equal in length to the line BD′, and in direction either coincident with, or opposite to, the chord AD: the locus of the point E will be an ellipsoid, with A for its centre, and with B for a point of its surface.

[*Elements of Quaternions*, Art. 217 (6). See also *Lectures*, Art. 465. If C is the centre of the sphere, the isosceles triangle ACD gives $\frac{CD}{DA} = K\frac{CA}{AD}$, or $CD = -AD^{-1}.CA.AD = -AE^{-1}.CA.AE$, and therefore

$$DB = CB + AE^{-1}.CA.AE = \iota + \rho^{-1}\kappa\rho$$

if $CB = \iota$, $AE = \rho$, $CA = \kappa$. By the property of the sphere $D'B \,.\, DB = CB^2 - CA^2 = \kappa^2 - \iota^2$, and by the construction $T\rho = TD'B = T(\iota^2 - \kappa^2) \,.\, TDB^{-1}$, or $T(\rho\iota + \kappa\rho) = T(\iota^2 - \kappa^2)$. Squaring both sides, we have $T\rho^2 T(\iota^2 + \kappa^2) + 2S\rho\iota K\kappa\rho = T(\iota^2 - \kappa^2)^2$, which reduces immediately to Hamilton's cyclic form.]

Ex. 7. Conceive two equal spheres to slide within two cylinders of revolution, whose axes intersect each other, in such a manner that the right line joining the centres of the spheres shall be parallel to a fixed right line; then the locus of the varying circle in which the two spheres intersect each other will be an ellipsoid, inscribed at once in both the cylinders.

[Hamilton, *Lectures*, Art. 496. Taking the spheres to be $T(\rho - t\alpha) = b$, $T(\rho - t\beta) = b$, where α, β and b are given and where t is a variable scalar, we find on elimination of t,

$$(\rho^2 + b^2)(\alpha^{-2} - \beta^{-2})(\alpha^2 - \beta^2) = 2S(\alpha^{-1} - \beta^{-1})\rho S(\alpha - \beta)\rho.]$$

ART. **78**. To find the right circular tangent cylinders of a quadric, observe that if the vertex of the tangent cone (Art 73 (VIII.)) passes off to infinity, the equation of the tangent cylinder parallel to α is

$$(S\rho\phi\rho + 1)S\alpha\phi\alpha - (S\rho\phi\alpha)^2 = 0. \quad \text{(I.)}$$

A right circular cylinder parallel to α and of radius $T\alpha^{-1}$ is represented by

$$TV\alpha\rho = 1, \quad \text{or} \quad (V\alpha\rho)^2 + 1 = 0, \quad \text{(II.)}$$

and identifying this with (I.) we have to satisfy

$$S\rho\phi\rho = \frac{(S\rho\phi\alpha)^2}{S\alpha\phi\alpha} + (V\alpha\rho)^2 \quad \text{(III.)}$$

for all vectors ρ, or what is equivalent we must identify

$$\phi\rho = \frac{\phi\alpha S\rho\phi\alpha}{S\alpha\phi\alpha} - \alpha V\alpha\rho. \quad \text{(IV.)}$$

This is identical for $\rho = \alpha$; and for $\rho = \phi\alpha$ we have

$$\phi^2\alpha = \frac{\phi\alpha S\alpha\phi^2\alpha}{S\alpha\phi\alpha} - \phi\alpha \,.\, \alpha^2 + \alpha S\alpha\phi\alpha. \quad \text{(V.)}$$

Here then is a linear relation connecting the vectors $\phi^2\alpha$, $\phi\alpha$ and α, and it follows (Art. 66) that α must be coplanar with a pair of axes, i and k suppose, and that (say)

$$\phi^2\alpha - (g_3 + g_1)\phi\alpha + g_3 g_1 \alpha = 0.$$

This gives on comparison with (V.)

$$S\alpha\phi\alpha = -g_3 g_1, \quad S\alpha\phi^2\alpha = (g_3 + g_1 + \alpha^2)S\alpha\phi\alpha, \quad S\alpha j = 0, \ldots \text{(VI.)}$$

and putting $\rho = j$ in the identity (IV.), we find

$$\alpha^2 = -g_2. \quad \text{(VII.)}$$

The identity is now satisfied for three non-coplanar vectors,

j, a and ϕa and therefore for all vectors; and if $U\beta = U\phi a$, the equation of the quadric is by (III.) reduced to

$$S\rho\phi\rho = b(S\rho U\beta)^2 + a(V\rho Ua)^2 = -1, \text{ ....(VIII.)}$$

where $$a = g_2, \quad b = g_2 - g_1 - g_3,$$

which is Hamilton's focal form, if we remark that by (VI.) and (VII.)

$$b = \frac{T(\phi a)^2}{Sa\phi a} = -\frac{Sa\phi^2 a}{Sa\phi a} = g_2 - g_1 - g_3.$$

If $a = ix + kz$ we have by (VI.) and (VII.)

$$x^2 + z^2 = g_2, \quad g_1x^2 + g_3z^2 = g_3g_1,$$

and $$a = i\sqrt{\frac{g_3(g_1 - g_2)}{g_1 - g_3}} + k\sqrt{\frac{g_1(g_2 - g_3)}{g_1 - g_3}}. \text{ ....(IX.)}$$

ART. **79**. *To find the generators of a quadric*, we express that when we substitute $\rho + ta$ in its equation, the equation is satisfied for all values of t. Thus

$$S\rho\phi\rho = -1, \quad S\rho\phi a = 0, \quad Sa\phi a = 0. \text{ ....(I.)}$$

From the second and third of these

$$Va\rho = x\phi a, \quad \text{or} \quad \rho = xa^{-1}\phi a + ya, \text{ ....(II.)}$$

and substituting for ρ in the equation of the quadric,

$$-1 = x^2 SVa^{-1}\phi a\phi Va^{-1}\phi a = x^2 mSVa^{-1}\phi aV\phi^{-1}a^{-1}a$$
$$= x^2m(Sa^{-1}aS\phi a\phi^{-1}a^{-1} - Sa^{-1}\phi^{-1}a^{-1}Sa\phi a),$$

or simply $x^2m = -1$. Thus the equation of the generator is

$$\rho = \pm\sqrt{-\frac{1}{m}}\,.\,a^{-1}\phi a + ya, \text{ ....(III.)}$$

it being implied by the form of this equation that $Sa^{-1}\phi a = 0$. Generators of one system correspond to the sign $+$, and those of the other system to the sign $-$.

Ex. 1. Prove that generators of opposite systems intersect.

Ex. 2. Find the locus of the feet of central perpendiculars on the generators.

[From the equation $\rho = \pm\sqrt{-\frac{1}{m}}a^{-1}\phi a$ we find $a \parallel V\rho\phi\rho$, and substitution in $Sa\phi a = 0$ gives a quartic cone which intersects the quadric along the locus.]

Ex. 3. Prove that the locus of intersections of generators which cut at right angles is the intersection of a sphere with the quadric.

[Note that a central plane parallel to a tangent plane cuts the asymptotic cone in lines parallel to the generators.]

Ex. 4. The locus of intersections of generators which cut at a given angle is

$$\tan u = 2\frac{\mathrm{T}\phi\rho\sqrt{-m}}{m\mathrm{T}\rho^2 - m'}; \quad \mathrm{S}\rho\phi\rho + 1 = 0.$$

[See Ex. 4, Art. 74.]

ART. **80.** When the equation of a quadric is given in the form

$$\mathrm{S}\rho\phi\rho - 2\mathrm{S}\epsilon\rho + l = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I.)}$$

in order to find its centre, or centres, we may replace the equation by

$$\mathrm{S}(\rho-\omega)\phi(\rho-\omega) + 2\mathrm{S}(\rho-\omega)(\phi\omega-\epsilon) + \mathrm{S}\omega\phi\omega - 2\mathrm{S}\epsilon\omega + l = 0, \ldots\text{(II.)}$$

and if ω terminates at a centre the part linear in $\rho-\omega$ vanishes, and ω is a solution of the equation

$$\phi\omega = \epsilon. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(III.)}$$

Operating by ψ we have

$$m\omega = \psi\epsilon, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(IV.)}$$

and the vector to the centre is finite and determinate if m is not zero. If m is zero and $\psi\epsilon$ not zero, the centre is at infinity in the direction of $\psi\epsilon$, and the surface is a paraboloid. If $\psi\epsilon$ is zero, m must also vanish, and the solution is (Art. 65)

$$m'\omega = \chi\epsilon + \psi\omega, \quad \psi\epsilon = 0, \ldots\ldots\ldots\ldots\ldots\ldots\text{(V.)}$$

and the surface has a line locus of centres and is a cylinder, $\psi\omega$ being parallel to the axis of ϕ corresponding to its zero root, and the length of $\psi\omega$ being indeterminate. If m' vanishes, the function ψ vanishes identically since ϕ is self-conjugate (Art. 67), and in fact ϕ is of the form $-ai\mathrm{S}i\rho$. If $\chi\epsilon$ is not zero, the line of centres is at infinity since (V.) can only be satisfied for infinite values of ω. If however $\chi\epsilon = 0$, the solution is

$$m''\omega = \epsilon + \chi\omega, \quad \chi\epsilon = 0, \ldots\ldots\ldots\ldots\ldots\ldots\text{(VI.)}$$

and the surface is a pair of parallel planes. More simply when

$$\phi\omega = -ai\mathrm{S}i\omega = \epsilon \quad \text{and} \quad \chi\epsilon = a\epsilon + ai\mathrm{S}i\epsilon = 0,$$

equation (III.) becomes $a\mathrm{S}i\omega = \mathrm{S}i\epsilon$.

In the case of the paraboloid, equation (V.) without the condition $\psi\epsilon = 0$, or

$$m'\omega = \chi\epsilon + uk, \quad \psi\epsilon \parallel k, \quad \phi k = 0 \ldots\ldots\ldots\ldots\text{(VII.)}$$

is the equation of the axis, remembering that $\psi\omega \parallel k = uk$ where u is an indeterminate scalar. We have in fact on operating by ϕ, $m'(\phi\omega - \epsilon) = -\psi\epsilon$, and the term linear in $\rho - \omega$ is proportional to $\mathrm{S}k(\rho-\omega)$. In like manner it may be shown that (VI.) without the condition $\chi\epsilon = 0$ represents the axial plane of a parabolic cylinder.

ART. **81**. We propose in this article to give a short account of the cone of the second degree and of sphero-conics. (See *Elements of Quaternions*, Art. 196.)

The equations

$$S\alpha\rho + 1 = 0, \quad S\frac{\beta}{\rho} - 1 = 0 \quad \text{(I.)}$$

represent respectively a plane and a sphere which passes through the origin of vectors. Combining these equations so as to eliminate $T\rho$, the equation

$$S\alpha\rho S\frac{\beta}{\rho} + 1 = 0, \quad \text{or} \quad S\alpha\rho S\beta\rho + \rho^2 = 0, \quad \text{or} \quad S\beta\rho S\frac{\alpha}{\rho} + 1 = 0, \quad \text{(II.)}$$

represents the cone whose vertex is the origin and which passes through the circle of intersection of the plane and sphere.

The third form of the equation shows that the cone passes through a second circle, the circle common to the plane and sphere

$$S\beta\rho + 1 = 0, \quad S\frac{\alpha}{\rho} - 1 = 0, \quad \text{(III.)}$$

and thus exhibits the theorem of Apollonius that an oblique cone having a circular base has a second series of circular sections.

The second form of the equation shows that the product of the cosines of the angles between an edge of the cone and the cyclic normals ($U\alpha$ and $U\beta$) is constant, for this is

$$SU.\alpha\rho SU.\beta\rho = T\alpha^{-1}\beta^{-1}; \quad \text{(IV.)}$$

or what is equivalent, if the cone is cut by a sphere concentric with the vertex, the product of the sines of the arcual perpendiculars let fall from any point of the sphero-conic of intersection on the two cyclic arcs (the great circles in the planes $S\alpha\rho = 0$, $S\beta\rho = 0$) is constant.

If $U\rho$ and $U\rho'$ are the vectors to any two points P and P′ on the sphero-conic, and if the great circle PP′ cuts the cyclic arcs in Q and Q′, it follows from the second of equations (II.) that $U(U\rho S\alpha U\rho' - U\rho' S\alpha U\rho)$ is the vector to one of the points (Q) and that $U(U\rho S\alpha U\rho - U\rho' S\alpha U\rho')$ is the vector to the second point (Q′), Q being in the cyclic plane $S\alpha\rho = 0$ and Q′ in $S\beta\rho = 0$. Hence, from the form of the expressions for the vectors to these points, we learn that the arc PQ is equal to the arc P′Q′.

If P′ and P″ are two fixed points on the sphero-conic, and if P is a variable point likewise on the conic; if the arcs PP′ and PP″ cut one cyclic arc ($S\alpha\rho = 0$) in Q′ and Q″, the length of the arc Q′Q″ is constant. This follows most easily by producing the radii of the points P, P′ and P″ to meet the plane $S\alpha\rho + 1 = 0$ of equation (I.) in the points P_0, P_0' and P_0''. It is evident that OQ′ and OQ″ are respectively parallel to P_0P_0' and P_0P_0'', and moreover the angle $P_0'P_0P_0''$ is constant since it is the angle subtended at a point on the circumference of a circle by two fixed points likewise on the circumference.

Given the cyclic arcs of a sphero-conic and a point on the conic, the conic is determined by elimination of t from the equations

$$S\alpha\rho S\beta\rho + t\rho^2 = 0, \quad S\alpha\gamma S\beta\gamma + t\gamma^2 = 0, \quad T\rho = 1,$$

the vector γ terminating at the given point, and for convenience the radius of the containing sphere being taken equal to unity.

The three propositions just proved are used by Hamilton to establish the associative principle of multiplication of quaternions. In the figure the great circles GLIM, CHBG, DAEC are the traces of the planes of three versors

$$q = \frac{OL}{OG}, \quad r = \frac{OH}{OC}, \quad s = \frac{OC}{OE}.$$

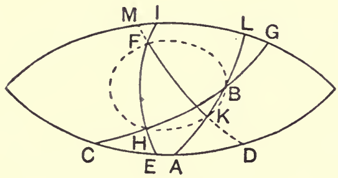

FIG. 25.

Constructing the product $rs = OH : OE$, the point H is determined and the sphero-conic HKBF is drawn through the point H having GLIM and DAEC for cyclic planes. Producing the arcs CH and EH, the points B, G, F and I are constructed. The point L is joined to B and LB is produced to K and A. The arc FK is drawn and produced to M and D. It follows then that the arcs GL and IM are equal and also the arcs CE and AD, and moreover $FM = DK$ and $AK = BL$ by the properties of the sphero-conic.

We have therefore

$$q \cdot rs = q \cdot \frac{OH}{OE} = q \cdot \frac{OI}{OF} = \frac{OM}{OI} \cdot \frac{OI}{OF} = \frac{OM}{OF} = \frac{OK}{OD} = \frac{OK}{OA} \cdot \frac{OA}{OD}$$

$$= \frac{OK}{OA} \cdot s = \frac{OL}{OB} \cdot s = \frac{OL}{OG} \cdot \frac{OG}{OB} \cdot s = qr \cdot s.$$

By proving the properties of the sphero-conic without employing the associative principle, this principle is established since we can show that for any three quaternions $q \cdot rs = qr \cdot s$.

In addition to the properties just proved for the sphero-conic, it is easy to see that great circle arcs which intersect at a point on the curve include supplemental arcs (such as CA and GL, Fig. 25) between the points in which they cut the cyclic arcs. Reciprocating these properties, the cyclic arcs become the foci E and F (Fig. 26) of the reciprocal sphero-conic, and if the two foci and one tangent arc AB are given, the conic can be constructed. If from any point on the sphere, two tangent arcs are drawn to the curve and also two focal arcs to the foci, then one focal arc makes with one tangent the same angle as the other focal arc makes with the other tangent. Moreover opposite arcs of a spherical quadrilateral, ABCD, circumscribing the conic subtend supplemental angles at the foci.

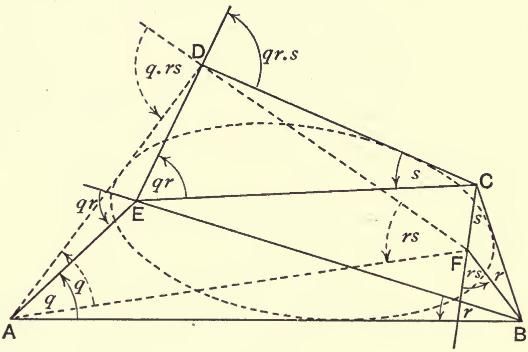

FIG. 26.

From these properties Hamilton deduces the associative principle. The versors q and r are represented by the directed angles BAE and EBA, and their product qr is (Art. 30, Ex. 5, p. 30) represented by the external angle at E or by the equal angle CED. A third versor s is represented by DCE, and the external angle of the triangle DEC represents the product $qr.s$ (namely, qr into s). Making FCB and CBF respectively equal to the angles of s and of r, the point F is found; and when the sphero-conic having E and F for foci and AB for tangent is constructed, it follows that BC and CD are also tangents on account of the equality of the angles marked r and of the angles marked s. Again, because CED was constructed equal to the supplement of AEB, the arc DA will be a tangent to the curve, and FAD will be equal to the angle of q, and DFA will be supplementary to CFB. Hence FAD and DFA represent respectively q and rs, and the external angle of the triangle ADF represents the product $q.rs$. But the angle between DA and DF is equal to the angle between DC and DF, and therefore $q.rs = qr.s$.

To find the locus of a point on the surface of a unit sphere, the sum of whose arcual distances from two fixed points, E *and* F, *is constant,* we have in the first place for the cosine of the sum of the arcs,

$$SU.\epsilon\rho SU\eta\rho - TVU.\epsilon\rho TVU.\eta\rho = \cos a; \quad \text{(V.)}$$

or on rationalization, we find the locus to be a sphero-conic,

$$(SU.\epsilon\rho)^2 + (SU.\eta\rho)^2 - 2\cos a\, SU.\epsilon\rho SU\eta\rho = \sin^2 a; \quad \text{(VI.)}$$

since $(SU.\epsilon\rho)^2 + (TVU.\epsilon\rho)^2 = 1$. (Compare *Elements of Quaternions*, Art. 360.)

This may also be written in the form

$$S(U\epsilon - \cos a\, U\eta)\rho = \pm \sin a\, TVU\eta\rho, \quad \text{(VII.)}$$

so that the sine of the arc between a point and a focus is proportional to the sine of the perpendicular on a directrix arc.

Many interesting examples and illustrations will be found in the *Elements*, Book II., Chap. III., Sections 1 and 2, and in Art. 306, and also in the sixth of the *Lectures on Quaternions*.

Ex. 1. Through three given points on the surface of a sphere, it is required to draw a sphero-conic so that a given great circle shall be one of its cyclic arcs.

[If γ_1, γ_2 and γ_3 are the vectors to the three given points, it is necessary to find β so that $S\beta\rho S\alpha\rho+\rho^2=0$ may be satisfied on replacing ρ by γ_1, γ_2 and γ_3, α being a given vector. The vector β is given by

$$\beta S\gamma_1\gamma_2\gamma_3=-\Sigma V\gamma_2\gamma_3(S\alpha\gamma_1{}^{-1})^{-1}.]$$

Ex. 2. Find the relations between the cyclic normals of a cone and its focal lines.

[Identifying (VI.) with the second form of (II.), the required relations are easily obtained.]

Ex. 3. Prove that

$$S.V.V\alpha\beta V\delta\epsilon V.V\beta\gamma V\epsilon\rho V.V\gamma\delta V\rho\alpha=0$$

represents the cone which has five edges parallel to five given vectors, α, β, γ, δ, ϵ, and show that the form of the equation furnishes a proof of Pascal's property of the hexagon inscribed to a conic. (*Lectures on Quaternions*, Art. 442.)

CONFOCAL QUADRICS.

ART. **82.** Quadrics of the family

$$S\rho(\phi+x)\rho=-1, \quad\ldots\ldots\ldots\ldots(\text{I.})$$

in which x is a variable parameter, are called concyclic, as they have common planes of circular section (Art. 77).

The reciprocal system of quadrics

$$S\rho(\phi+x)^{-1}\rho=-1 \quad\ldots\ldots\ldots\ldots(\text{II.})$$

is called a confocal system.

Because we may write (II.) in the form

$$S\rho(\psi+x\chi+x^2)\rho=-(m+m'x+m''x^2+x^3), \quad\ldots\ldots\ldots(\text{III.})$$

it appears that three quadrics (II.) pass through an arbitrary point; and reciprocally, three quadrics (I.) touch an arbitrary plane. Also one quadric (I.) passes through an arbitrary point, and one quadric (II.) touches an arbitrary plane.

Confocal quadrics cut at right angles. Let x, y and z be the parameters of the three quadrics which pass through an arbitrary point (α). Then

$$0=S\alpha(\phi+x)^{-1}\alpha-S\alpha(\phi+y)^{-1}\alpha=S\alpha[(\phi+x)^{-1}-(\phi+y)^{-1}]\alpha$$

$$=S\alpha.\frac{(\phi+y)-(\phi+x)}{(\phi+x)(\phi+y)}\alpha=(y-x)S\alpha(\phi+x)^{-1}(\phi+y)^{-1}\alpha,$$

for all functions $\phi+x$, $(\phi+x)^{-1}$, etc., have the same axes, and are therefore commutative (Art. 66, Ex. 2, p. 95). Thus at any point of the intersection of the quadrics x and y,

$$S\rho(\phi+x)^{-1}(\phi+y)^{-1}\rho=0\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(IV.)}$$

which expresses that the normals (Art. 73, p. 108) $(\phi+x)^{-1}\rho$ and $(\phi+y)^{-1}\rho$ are at right angles.

Ex. 1. Reduce $S\rho\,.\,\dfrac{\phi^2+u\phi+v}{(\phi+x)(\phi+y)(\phi+z)}\,.\,\rho$ to a sum of the form

$$AS\rho(\phi+x)^{-1}\rho+BS\rho(\phi+y)^{-1}\rho+CS\rho(\phi+z)^{-1}\rho.$$

[We may employ the method of partial fractions, and treat ϕ as a scalar, it being commutative with scalars and with $\phi+x$, etc.]

Ex. 2. If x, y and z are the parameters of the confocals through the extremity of the vector ρ, the expressions

$$S\rho\,.\,(\phi+x)^{-1}(\phi+y)^{-2}\rho,\quad S\rho(\phi+x)^{-1}(\phi+y)^{-1}(\phi+z)^{-1}\rho,\quad S\rho(\phi+x)^{-1}(\phi+z)^{-2}\rho,$$

are respectively equal to

$$\frac{1}{y-x}T(\phi+y)^{-1}\rho^2,\quad \text{zero,}\quad \text{and}\quad \frac{1}{z-x}T(\phi+z)^{-1}\rho^2.$$

Ex. 3. Prove that

$$\sqrt{x-y}\,.\,U(\phi+y)^{-1}\rho,\quad \sqrt{x-z}\,.\,U(\phi+z)^{-1}\rho$$

are the principal axes of the central section of the quadric x made by the plane parallel to the tangent plane at ρ.

Ex. 4. Find the centres of curvature at a point on the quadric x, and prove that they are the poles of the tangent plane to x with respect to the confocals y and z.

[If γ is the vector to a centre of curvature, two consecutive normals intersect at its extremity, or $\gamma=\rho+t(\phi+x)^{-1}\rho$ is stationary when ρ and t vary. Therefore

$$[1+t(\phi+x)^{-1}]d\rho+(\phi+x)^{-1}\rho dt=0,\quad \text{or}\quad (\phi+x+t)d\rho+\rho dt=0,$$

or

$$d\rho+(\phi+x+t)^{-1}\rho dt=0.$$

Operate with $S(\phi+x)^{-1}\rho$, and $S\rho(\phi+x)^{-1}(\phi+x+t)^{-1}\rho=0$, and on comparison with (IV.) the roots of this quadratic in t are seen to be $y-x$ and $z-x$. Therefore $\gamma=(\phi+y)(\phi+x)^{-1}\rho$, $\gamma'=(\phi+z)(\phi+x)^{-1}\rho$ are the vectors to the two centres. Observe that $d\rho$ is also tangential to the quadric z. Compare Art. 87, Ex. 1, p. 136, for the method employed.]

Ex. 5. If x, y and z are the parameters of the three confocals through the extremity of the vector ρ, prove that

$$x+y+z=-m''-\rho^2\,;\quad yz+zx+xy=m'+S\rho\chi\rho\,;\quad xyz=-m-S\rho\psi\rho.$$

Ex. 6. Prove that the plane $S\lambda\rho+1=0$ touches a confocal at the extremity of the vector

$$\rho=\lambda^{-1}(V\lambda\phi\lambda-1)\,;$$

and show that the locus of points of contact for a system of parallel planes is a rectangular hyperbola.

Ex. 7. Prove that the locus of points of contact of planes through a line is a twisted cubic.

[Put for λ in the last example $(\lambda+t\mu)(1+t)^{-1}$ and verify that an arbitrary plane meets the curve in three points.]

Ex. 8. The locus of the poles of a plane with respect to a system of confocals is a right line.

Ex. 9. The locus of the poles of planes through a given line is a hyperbolic paraboloid.

[$\rho=(\phi+u)(\lambda+t\mu)(1+t)^{-1}$ is the locus of a line dividing two given lines similarly.]

Ex. 10. The plane $\quad S\rho\lambda\phi\lambda=0$

is the locus of poles of planes perpendicular to λ.

ART. 83. In many investigations relating to the confocals through a given point, the extremity of the vector α, it is convenient to employ the vectors

$$\lambda=(\phi+x)^{-1}\alpha,\quad \mu=(\phi+y)^{-1}\alpha,\quad \nu=(\phi+z)^{-1}\alpha, \quad\text{(I.)}$$

which when originating at the centre terminate at the reciprocals of the three tangent planes. These vectors are of course normal to the three confocals. We have then

$$\alpha=(\phi+x)\lambda=(\phi+y)\mu=(\phi+z)\nu,\quad S\lambda\alpha=S\mu\alpha=S\nu\alpha=-1; \quad\text{(II.)}$$

and because these equations give

$$-1=S\mu(\phi+x)\lambda=S\lambda(\phi+y)\mu,\quad\text{or}\quad (x-y)S\lambda\mu=0,$$

it follows that

$$S\mu\nu=S\nu\lambda=S\lambda\mu=0, \quad\text{(III.)}$$

or confocals cut at right angles.

We also have from the same equations

$$\lambda=\mu+(y-x)(\phi+x)^{-1}\mu,\ \text{etc.}, \quad\text{(IV.)}$$

so that $\quad \mu^2+(y-x)S\mu(\phi+x)^{-1}\mu=0,\quad (y-x)S\nu(\phi+x)^{-1}\mu=0,$

or $\quad (x-y)^{-1}=-SU\mu(\phi+x)^{-1}U\mu=+SU\lambda(\phi+y)^{-1}U\lambda,\ \text{etc.},$

$$S\mu(\phi+x)^{-1}\nu=0,\ \text{etc.} \quad\text{(V.)}$$

And the axes of the section of the quadric x parallel to the tangent plane are $\sqrt{x-y}\,.\,U\mu$, $\sqrt{x-z}\,.\,U\nu$; and those of the section of the quadric y parallel to its tangent plane are $\sqrt{y-x}\,.\,U\lambda$, $\sqrt{y-z}\,.\,U\nu$.

Introducing a new self-conjugate function θ defined by the equation

$$\theta\rho=\phi\rho+\alpha S\alpha\rho, \quad\text{(VI.)}$$

we may replace equations (II.) by

$$(\theta+x)\lambda=(\theta+y)\mu=(\theta+z)\nu=0, \quad\text{(VII.)}$$

so that λ, μ and ν are the axes and x, y and z the roots of this function.

If $S\omega\rho=-1$ is the equation of any plane through the point α, and if ϖ is the pole of the plane with respect to any confocal u,

$$\varpi=(\phi+u)\omega,\quad\text{or}\quad \varpi-\alpha=(\theta+u)\omega, \quad\text{(VIII.)}$$

because $-\alpha=+\alpha S\alpha\omega$. If the plane *touches* the quadric u, the pole lies in the plane, and the vector $\varpi-\alpha$ (joining two points in the plane) is normal to ω. Thus in order to determine the point of contact of the plane

$S\omega\rho = -1$ and the parameter of the touched quadric, it is only necessary to operate on ω by the function θ and to resolve $\theta\omega$ along and at right angles to ω; for

$$\theta\omega = \omega V\omega^{-1}\theta\omega + \omega S\omega^{-1}\theta\omega = \varpi - \alpha - u\omega;\ \varpi = \alpha + \omega V\omega^{-1}\theta\omega,\ \ u = -S\omega^{-1}\theta\omega. \quad \text{(IX.)}$$

The vector ϖ being still supposed to terminate in the plane, the vector $\varpi - \alpha (=\tau)$ is tangential to the surface u and perpendicular to ω. Hence as ϖ varies subject to the condition $S\varpi\omega = S\alpha\omega = -1$, we find by (VIII.) that

$$S(\varpi - \alpha)(\theta + u)^{-1}(\varpi - \alpha) = 0, \quad \text{or} \quad S\tau(\theta + u)^{-1}\tau = 0 \ \ldots\ldots\ldots\ldots \text{(X.)}$$

is the equation of the *tangent cone* from α to the confocal u, referred in the first case to the centre of the quadrics and in the second to the extremity of α. The form of the equations shows that the tangent cones drawn from a point are confocal. They intersect in pairs along any line through the point, for (X.) may be replaced by

$$S\tau(\psi_\theta + u\chi_\theta + u^2)\tau = 0, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(XI.)}$$

and may be regarded as a quadratic determining the quadrics touched by a given line ($U\tau = \text{const.}$); and they intersect at right angles by the general property of confocals.

We can thus determine the two quadrics touched by an arbitrary line.

Ex. 1. Prove that

$$(\psi_\theta + u\chi_\theta + u^2)\rho = (\psi + u\chi + u^2)\rho + V\alpha(\phi + u)V\alpha\rho.$$

Ex. 2. A right line defined by the vectors σ and τ of Art. 36, Ex. 4, touches the confocals whose parameters are the roots of the equation,

$$S\tau(\psi + u\chi + u^2)\tau - S\sigma(\phi + u)\sigma = 0.$$

Ex. 3. The lines through a given point touching confocals with a given sum of parameters, generate the reciprocal of the tangent cone to a fixed confocal.

[The cone of the lines is $S\tau(\theta - m'' - \alpha^2 - v)\tau = 0$, if v is the sum of the parameters.]

Ex. 4. If ν and ν' are the vectors to the reciprocals of the tangent planes of the confocals u and u' at the points A and B, and if τ is the vector AB,

$$S\tau(\nu + \nu') = (u' - u)S\nu\nu'.$$

[Here $\tau = (\phi + u')\nu' - (\phi + u)\nu$. This is Gilbert's theorem.]

Ex. 5. If the points A and B are both points of contact of the line with the quadrics,

$$S\nu\nu' = 0, \quad S\nu\phi\nu' + 1 = 0.$$

ART. **84.** There is a third general method which is often useful for dealing with the properties of confocals. Writing the equations of the three confocals through a point in the forms

$$T(\phi + x)^{\frac{1}{2}}\rho = 1, \quad T(\phi + y)^{\frac{1}{2}}\rho = 1, \quad T(\phi + z)^{\frac{1}{2}}\rho = 1, \ \ldots\ldots\ldots\ldots \text{(I.)}$$

we are led to assume

$$\rho = \sqrt{\{(\phi + x)(\phi + y)(\phi + z)\}}\,\epsilon \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(II.)}$$

as an expression for the vector to the point of intersection. The square roots $(\phi + x)^{\frac{1}{2}}$, etc., are commutative, and, accordingly, on substitution in

$$S\rho(\phi + x)^{-1}\rho = -1,$$

we find $$-1 = S\epsilon(\phi + y)(\phi + z)\epsilon = S\epsilon\phi^2\epsilon + (y + z)S\epsilon\phi\epsilon + yz\epsilon^2. \ \ldots\ldots\ldots \text{(III.)}$$

This is identically satisfied, for the confocal x as well as for the confocals y and z, if

$$\epsilon^2=0,\quad S\epsilon\phi\epsilon=0,\quad S\epsilon\phi^2\epsilon=-1\,;\quad\text{(IV.)}$$

or what is equivalent, if

$$\epsilon^2=0,\quad S\epsilon\chi\epsilon=0,\quad S\epsilon\psi\epsilon=-1\,;\quad\text{(V.)}$$

that is, if ϵ is the vector to a point of intersection of three known surfaces, one of which is of course imaginary. Therefore (II.) coupled with the conditions (IV.) or (V.) is the vector to a point of intersection of the three confocals; and allowing any two of the parameters, y and z, in (II.) to vary, the vector equation represents the surface x; if only one parameter (x) varies, the equation represents the curve of intersection of the confocals y and z.

Again, we may differentiate ρ, regarded as a function of x, y and z, as given by equation (II.) just as if ϕ were a scalar, and we have

$$d\rho=\tfrac{1}{2}\cdot\left(\frac{dx}{\phi+x}+\frac{dy}{\phi+y}+\frac{dz}{\phi+z}\right)\rho\,;\quad\text{(VI.)}$$

and the method easily lends itself to the treatment of lines traced on a quadric surface.

Ex. 1. Prove that the vectors $(\phi+x)^{-1}\rho$, $(\phi+y)^{-1}\rho$, $(\phi+z)^{-1}\rho$ are mutually rectangular, and that the squares of their tensors are

$$\frac{(z-x)(x-y)}{m(x)},\quad \frac{(x-y)(y-z)}{m(y)},\quad \frac{(y-z)(z-x)}{m(z)},$$

where $m(x)=m+m'x+m''x^2+x^3$, and where x, y and z are the parameters of the confocals through the extremity of ρ.

[Using (II.), we have $S\rho(\phi+y)^{-1}(\phi+z)^{-1}\rho=S\epsilon(\phi+x)\epsilon=0$. Also

$$S\rho(\phi+x)^{-2}\rho=S\epsilon(\phi+x)^{-1}(\phi+y)(\phi+z)\epsilon.$$

This is reduced by replacing y by $x+y-x$, etc., to $S\epsilon(\phi+x)^{-1}\epsilon$ multiplied by a factor. On inversion of $(\phi+x)^{-1}$ the rest follows.]

Ex. 2. Find $T\rho^2$ in terms of x, y and z.

$$[x+y+z+m''=T\rho^2.]$$

Ex. 3. Express the vector ϵ in terms of the roots and axes of ϕ.

Ex. 4. Prove that

$$Td\rho^2=\tfrac{1}{4}\Sigma\frac{(z-x)(x-y)}{m(x)}dx^2.$$

Ex. 5. Prove that $\rho=(\phi+u)(\phi+x)^{-\frac{1}{2}}(\phi+y)^{\frac{1}{2}}(\phi+z)^{\frac{1}{2}}\epsilon$ is the equation of a tangent to the curve of intersection of the quadrics y and z; u being alone variable.

[Use (VI.).]

Ex. 6. Prove that $\rho=(\phi+x)^{-\frac{1}{2}}(\phi+y)^{\frac{3}{2}}(\phi+z)^{\frac{1}{2}}\epsilon$ is the equation of the surface of centres of the quadric x—the locus of the principal centres of curvature—when y and z vary. (See Art. 82, Ex. 4.)

Ex. 7. Find the lengths of the principal radii of curvature in terms of x, y and z.

Ex. 8. The imaginary right line, t variable,

$$\rho=(\phi+t)(\phi+x)^{\frac{1}{2}}\epsilon$$

is an umbilical generator of the quadric x.

[It is evidently a generator of the quadric, and parallel to a line to a circular point at infinity for $T(\phi+x)^{\frac{1}{2}}\epsilon=0$. That is, it is one of the eight generators through the four points in which the imaginary circle at infinity cuts the quadric. But the tangent plane at an umbilic cuts the surface in a point circle—or a pair of these imaginary generators. See Art. 67, Ex. 1, p. 96.]

Ex. 9. Find the locus of a point through which two of the three intersecting confocals coincide. Show that it is a developable surface generated by the tangent lines to the curve

$$\rho=(\phi+x)^{\frac{3}{2}}\epsilon.$$

[This is the locus of the umbilical generators of the system, or the circumscribing developable.]

Ex. 10. The focal conics are double curves on this developable.

[Put t equal $-g_1$, $-g_2$ or $-g_3$ in the equation of Ex. 8, and we get a plane curve in one of the principal planes. For $t=-g_1$ we have

$$S\rho(\phi-g_1)^{-1}\rho=S\epsilon(\phi-g_1)(\phi+x)\epsilon=-1,\quad Si\rho=0.$$

The conic is double on the developable because a double sign is lost owing to the destruction of the component of the vector normal to the plane.]

Ex. 11. If α is a constant vector, and x, y variable scalars, the equation

$$\rho=(\phi^2+x\phi+y)^{\frac{1}{2}}\alpha$$

represents a quadric surface, ϕ being a self-conjugate function.

[Assume the equation of the quadric to be $S\rho(a\phi^2+b\phi+c)\rho+1=0$, and determine the constants a, b and c.]

Ex. 12. Prove that the imaginary vector ϵ of equation (IV.) satisfies the relation $\sqrt{-1}\,.\,\epsilon=V\epsilon\phi\epsilon$.

EXAMPLES TO CHAPTER IX.

Ex. 1. Three right lines through a common point are mutually at right angles. If the first and second move in the planes $S\lambda\rho=0$ and $S\mu\rho=0$ respectively, the third describes the cone

$$SV\lambda\rho V\mu\rho=0.$$

Ex. 2. The cone

$$\frac{S\alpha iS\alpha jS\alpha k}{S\alpha\rho}+\frac{S\beta iS\beta jS\beta k}{S\beta\rho}+\frac{S\gamma iS\gamma jS\gamma k}{S\gamma\rho}=0$$

contains the six unit vectors i, j, k and α, β, γ, the vectors of each set being mutually perpendicular.

Ex. 3. If the cone $S\rho\phi\rho=0$ has three mutually rectangular edges, the condition $m''=0$ must be satisfied; if it touches three mutually rectangular planes, $m'=0$.

Ex. 4. The four cones of revolution which touch the planes

$$S\lambda\rho=0,\quad S\mu\rho=0,\quad S\gamma\rho=0$$

are represented by $T\,.\,V\rho^{-1}V\rho\Sigma\pm V\mu\nu T\lambda(S\lambda\mu\nu)^{-1}=1$;

and the cones of revolution through the three lines

$$V\lambda\rho=0,\quad V\mu\rho=0,\quad V\nu\rho=0$$

are represented by $T\,.\,\rho^{-1}S\rho\Sigma\pm V\mu\nu T\lambda(S\lambda\mu\nu)^{-1}=1.$

Ex. 5. Three points fixed on a line move in given planes. Find the locus of a fourth point fixed on the line, and show that it is represented by an equation of the form

$$\mathrm{T}(a\mathrm{V}\mu\nu\mathrm{S}\lambda\rho+b\mathrm{V}\nu\lambda\mathrm{S}\mu\rho+c\mathrm{V}\lambda\mu\mathrm{S}\nu\rho)=1.$$

Ex. 6. Interpret the equation

$$\mathrm{T}\beta^{-1}\mathrm{V}\beta\rho=e\mathrm{T}\lambda^{-1}\mathrm{S}\lambda\rho$$

as determining the locus of a point moving in accordance with a certain law in relation to a given line and a given plane.

Ex. 7. The polar planes of points situated on certain fixed lines cut a quadric in circles.

Ex. 8. Find the locus of the centre of a sphere which rolls along two straight wires.

Ex. 9. Determine the locus of the vertex of a right cone standing on a given ellipse of which α and β are the principal vector radii.

Ex. 10. A plane cuts a constant volume from a pyramid having its vertex at the centre of a quadric. Find the locus of the pole of the plane with respect to the quadric.

Ex. 11. Find a tangent plane to a quadric which along with three mutually conjugate planes passing through the centre forms a tetrahedron of minimum volume.

Ex. 12. Find the locus of the point of intersection of three mutually perpendicular planes each of which touches one of three given confocal quadrics.

Ex. 13. Find the locus of the foot of the central perpendicular on a plane through the extremities of three mutually conjugate radii of a quadric.

Ex. 14. Find the locus of intersection of tangent planes at the extremities of three mutually conjugate radii of a quadric.

Ex. 15. Find the locus of a point whence three mutually perpendicular tangent lines can be drawn to a quadric.

Ex. 16. Find the locus of a point whence three tangent lines can be drawn to a quadric so as to be parallel to three mutually conjugate radii.

Ex. 17. Show that the equation

$$\frac{2\phi\rho}{\mathrm{S}\rho\phi\rho}-\frac{\rho}{\rho^2}-\frac{\phi^2\rho}{(\phi\rho)^2}=0$$

determines the directions of the radii of the quadric $\mathrm{S}\rho\phi\rho+1=0$ which are most or least inclined to the corresponding normals. Solve this equation.

Ex. 18. Through the extremity of the vector α mutually perpendicular lines are drawn to cut a quadric. Prove that

$$\frac{-m''}{1+\mathrm{S}\alpha\phi\alpha}=\frac{1}{x_1x_2}+\frac{1}{y_1y_2}+\frac{1}{z_1z_2}$$

where x_1 and x_2 are the intercepts on one of the lines.

Ex. 19. From a point on the quadric $\mathrm{S}\rho\phi\rho+1=0$, the extremity of the vector α, mutually rectangular lines are drawn to terminate on the surface. The plane through their extremities passes through the extremity of the vector

$$\alpha-\frac{2\phi\alpha}{m''}.$$

Ex. 20. Find the volume of the frustum of the cone whose vertex is at the centre of the quadric $S\rho\phi\rho+1=0$ and whose base is the intersection of the quadric with the plane $S\lambda\rho+1=0$.

Ex. 21. If UVq is a fixed vector γ, eliminate the scalar t and the variable part of q from the relation

$$\rho=q(\beta+t\alpha)q^{-1}$$

and discuss the locus represented by

$$T\rho=T\left(\beta+\alpha\frac{S\gamma(\rho-\beta)}{S\gamma\alpha}\right).$$

Ex. 22. The vectors α, β and γ being unit and mutually rectangular, show that the condition that

$$T\phi\alpha+T\phi\beta+T\phi\gamma$$

should be a maximum or minimum is

$$V\alpha\phi'U\phi\alpha+V\beta\phi'U\phi\beta+V\gamma\phi'U\phi\gamma=0$$

where ϕ is an arbitrary vector function, and prove that this is equivalent to

$$T\phi\alpha=T\phi\beta=T\phi\gamma.$$

(a) Hence derive a theorem concerning the conjugate radii of an ellipsoid.

Ex. 23. Through a variable point Q on a fixed line $V(\rho-\beta)\alpha=0$, a plane is drawn perpendicular to a fixed line (γ). Find the locus of points P in the variable plane for which $\mathrm{TOP}=e\mathrm{TPQ}$ where e is a given scalar.

Ex 24. Show that the section of the cone $S\rho\phi\rho=0$ by the plane $S\lambda\rho+1=0$ is equal to the section of the quadric $S\rho\phi\rho S\lambda\phi^{-1}\lambda+1=0$ by the plane $S\lambda\rho=0$.

Ex. 25. Find the equation of the surface which is generated by transversals of the lines $V(\rho-\beta)\alpha=0$, $V(\rho-\beta')\alpha'=0$ and of the ellipse

$$\rho=\gamma+\gamma'\cos t+\gamma''\sin t.$$

Ex. 26. The envelope of the planes of intersection of the sphere $2S\lambda\rho^{-1}=1$ with a variable sphere passing through the origin and having its centre on the quadric $S\rho\phi\rho+1=0$ is the cone

$$(S\lambda\rho)^2+S\rho\phi^{-1}\rho=0.$$

Ex. 27. From the extremity of the vector δ which terminates on the quadric $S\rho\phi\rho+1=0$, a right line is drawn to intersect the vector radius α, one of three mutually conjugate radii α, β, γ, and to be parallel to the plane containing the other two. It meets the ellipsoid again at the extremity of the vector $-\delta-2\alpha S\delta\phi\alpha$; and the plane $S\lambda\rho+1=0$ which passes through the three points thus determined by the three radii is given by

$$\lambda=\left(\frac{\phi\alpha}{S\delta\phi\alpha}+\frac{\phi\beta}{S\delta\phi\beta}+\frac{\phi\gamma}{S\delta\phi\gamma}\right).$$

Ex. 28. Show that

$$\rho=-\frac{\phi^{-1}(\lambda+t\lambda')(1+t)}{S(\lambda+t\lambda')\phi^{-1}(\lambda+t\lambda')}$$

is the locus of the centres of sections of the quadric $S\rho\phi\rho+1=0$ made by planes through the intersection of the planes $S\lambda\rho+1=0$, $S\lambda'\rho+1=0$; and discuss the nature of the curve.

Ex. 29. Show that the surface represented by the equation

$$S[\lambda(S\lambda'\rho+1)-\lambda'(S\lambda\rho+1)][\mu(S\mu'\rho+1)-\mu'(S\mu\rho+1)]=0$$

may be generated by the intersection of two perpendicular planes each of which contains a fixed line.

Ex. 30. Prove that the foci of central sections of the quadric $S\rho\phi\rho+1=0$ generate the surface

$$\frac{\rho^2}{S\rho\phi\rho}-\frac{(V\rho\phi\rho)^2}{SV\rho\phi\rho\phi V\rho\phi\rho}+\rho^2=0.$$

Ex. 31. The envelope of a sphere which passes through the centre of a quadric and which cuts it in a pair of circles is a quartic surface touching the quadric along a sphero-conic.

Ex. 32. Quadrics similar to $S\rho\theta\rho+1=0$ are described on a system of parallel chords of $S\rho\phi\rho+1=0$ as diameters. Prove that the envelope of these quadrics is also a quadric, and find its equation.

Ex. 33. Prove that
$$\Sigma\rho_n{}^2-S\alpha\Sigma\rho_n+\frac{2m'}{m}=0$$
where ρ_n is the vector to the foot of a normal from the extremity of the vector α to the surface $S\rho\phi\rho+1=0$ and where m' and m are the second and third invariants of the function ϕ.

Ex. 34. If a right line cuts a quadric at the angles θ and θ', show that

$$\frac{\sin\theta}{p}=\frac{\sin\theta'}{p'}$$

where p and p' are the central perpendiculars on the tangent planes at the points of intersection.

Ex. 35. If n is the length of the chord which is normal to a quadric at the extremity of ρ,

$$\frac{2}{n}=m''p-(m'-mT\rho^2).p^3.$$

Ex. 36. Pairs of mutually rectangular tangent planes are drawn through the extremity of the vector α to the quadric surface $S\rho\phi\rho+1=0$; prove that the locus of their intersection is

$$1+S\frac{\alpha-\rho}{V\alpha\rho}\cdot\phi^{-1}\frac{\alpha-\rho}{V\alpha\rho}=(\alpha-\rho)^2S\frac{1}{V\alpha\rho}\phi^{-1}\cdot\frac{1}{V\alpha\rho},$$

and show that this equation may be reduced to

$$m(V\alpha\rho)^2+S(\alpha-\rho)\psi(\alpha-\rho)=m'(\alpha-\rho)^2.$$

Ex. 37. The sum of the products of the perpendiculars from the two extremities of three mutually conjugate diameters on any tangent plane to a quadric is twice the square of the central perpendicular on the tangent plane.

Ex. 38. In terms of the vectors $\tau=\rho_2-\rho_1$, $\sigma=V\rho_1\rho_2$, show that the equation
$$S\sigma\psi\tau=0$$
represents the chords of the quadric $S\rho\phi\rho+1=0$ which enjoy the property that the normals at their extremities intersect.

Ex. 39. The locus of the centres of chords at whose extremities the normals intersect and which are parallel to a fixed direction τ is the right line
$$S\rho\phi\tau=0,\ S\rho\tau\psi\tau=0.$$

Ex. 40. Prove that the squared radii of the circular sections of the quadric $g\rho^2+2S\lambda\rho S\mu\rho+1=0$ which pass through the extremity of the vector α are

$$g^{-1}+\lambda^{-2}(S\lambda\alpha)^2 g'g''g^{-2} \text{ and } g^{-1}+\mu^{-2}(S\mu\alpha)^2 g'g''g^{-2}$$

where g, g' and g'' are the latent roots of the linear function determining the quadric. Interpret these results.

Ex. 41. Determine the spheres cut in diametral planes by a quadric.

Ex. 42. If planes through an edge (ρ) of the cone $S\rho\phi\rho=0$ and through the vectors α and β respectively meet the cone again in edges coplanar with the vector γ, show that

$$S(\rho S\alpha\phi\alpha-2\alpha S\rho\phi\alpha)(\rho S\beta\phi\beta-2\beta S\rho\phi\beta)\gamma=0,$$

and reduce this by the aid of the equation of the cone to

$$S\rho\phi\alpha S\beta\phi\gamma+S\rho\phi\beta S\gamma\phi\alpha-S\rho\phi\gamma S\alpha\phi\beta=0.$$

Ex 43. Using the notation of Art. 38, p. 42, show that if a translation represented by the vector ω will carry the tetrahedron ABCD so that it becomes inscribed to the quadric $S\rho\phi\rho+1=0$, we shall have

$$\omega=\tfrac{1}{2}v^{-1}\phi^{-1}\Sigma\lambda S\alpha\phi\alpha\,;\quad S\Sigma\lambda S\alpha\phi\alpha\phi^{-1}\Sigma\lambda S\alpha\phi\alpha+4v\Sigma l S\alpha\phi\alpha+4v^2=0.$$

Ex. 44. It is required to place a pair of tetrahedra ABCD and A′B′C′D′ so that their vertices may be corresponding points on a pair of confocal quadrics. (Robert Russell.)

(a) A quaternion statement of this problem is to determine a self-conjugate function Φ, a scalar u, a quaternion q and a pair of vectors κ and κ' so that the conditions

$$\Phi^{-\frac{1}{2}}(\rho-\kappa)=(\Phi+u)^{-\frac{1}{2}}(q\rho'q^{-1}-\kappa')=a \textit{ unit vector}$$

may be satisfied when ρ and ρ' terminate at corresponding vertices of the tetrahedra in their initial positions.

(b) If ϕ is the linear vector function defined by the relations

$$\phi(\alpha-\delta)=\alpha'-\delta',\quad \phi(\beta-\delta)=\beta'-\delta',\quad \phi(\gamma-\delta)=\gamma'-\delta',$$

we find that $\quad u\Phi^{-1}=\phi'\phi-1,$ and $q(\ )q^{-1}=(\phi'\phi)^{-\frac{1}{2}}\phi'.$

(c) Also in the notation of Art. 38, κ and u are given by

$$2v\,.\,(\phi'\phi-1)\kappa=-\Sigma\lambda S\alpha(\phi'\phi-1)\alpha,\quad v(u+S\kappa(\phi'\phi-1)\kappa)+\Sigma l S\alpha(\phi'\phi-1)\alpha=0.$$

Ex. 45. A plane mirror (normal ν) is moved so as to reflect the light from a star in a fixed direction (δ). Show that if γ is the unit vector towards the celestial pole, σ the unit vector towards the star at the time $t=0$, the vector ν must describe the cone represented by

$$\nu\,||\,(\gamma^{-\frac{2t}{\pi}}\sigma\gamma^{\frac{2t}{\pi}}+\delta)\quad\text{or}\quad \nu^2 S\gamma(\sigma+\delta)=2S\nu\gamma S\nu\delta.$$

(a) Show that the vector

$$\gamma^{-\frac{t}{\pi}}\lambda\gamma^{\frac{t}{\pi}}\,.\,\gamma^{-\frac{2t}{\pi}}\sigma\gamma^{\frac{2t}{\pi}}\,.\,\gamma^{-\frac{t}{\pi}}\lambda\gamma^{\frac{t}{\pi}}$$

is independent of t provided the vector λ satisfied a certain condition of perpendicularity, and interpret.

CHAPTER X.

GEOMETRY OF CURVES AND SURFACES.

(i) *Metrical Properties of Curves.*

ART **85.** Supposing that from each point of a curve a vector η is drawn, variable with the position of the point, let us consider the rate of rotation requisite to produce the change of direction of the vectors η as we pass along the curve. In the figure P and P$'$ are any two points on the curve, and the vector $\mathrm{PH}=\mathrm{U}\eta$ is a unit vector along the *emanant* vector η drawn from P, while $\mathrm{P'H'}=\mathrm{U}\eta'$ is a unit vector along the emanant η' drawn from P$'$. The vector PH$''$ is drawn equal to P$'$H$'$.

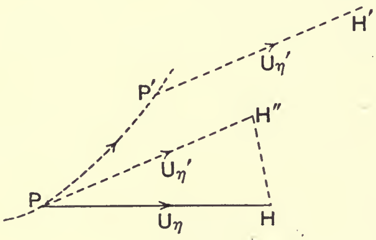

FIG. 27.

In the limit the quaternion

$$\frac{\mathrm{U}\eta'-\mathrm{U}\eta}{\mathrm{U}\eta\mathrm{T}(\rho'-\rho)}=\frac{\mathrm{HH''}}{\mathrm{PH}\,.\,\mathrm{TPP'}} \quad \text{..........................(I.)}$$

is a vector perpendicular to η and to η' so that rotation round it from η to η' is positive, the angle of the quaternion (the exterior angle at H) being ultimately equal to a right angle. The tensor of this vector is ultimately equal to the ratio of the circular measure of the angle HPH$''$ (the angle between η and η') to the arc of the curve, and thus the vector represents in magnitude and direction the rate of rotation in question. In terms of the differential of $U\eta$ and the corresponding differential of $\rho(=\text{OP})$, the vector of rotation is

$$\iota=\frac{dU\eta}{U\eta Td\rho}=V\frac{d\eta}{\eta}\cdot\frac{1}{Td\rho}=\frac{V\eta d\eta}{T\eta^2 Td\rho}, \qquad\text{(II.)}$$

the second form of the expression for the vector being deduced from (IV.), Art. 53, p. 68, and the third form resulting from the consideration that

$$V\alpha\beta^{-1}=V\alpha\beta\,.\,\beta^{-2}=-V\alpha\beta\,.\,T\beta^{-2}=+V\beta\alpha\,.\,T\beta^{-2}.$$

If, in particular, we replace the vector η by $d\rho$, a vector tangential to the curve, we have for the vector of rotation of the tangent, or the *vector curvature* at P,

$$\frac{dUd\rho}{d\rho}=V\frac{d^2\rho}{d\rho}\cdot\frac{1}{Td\rho}=\frac{Vd\rho d^2\rho}{Td\rho^3}, \qquad\text{(III.)}$$

for in accordance with the foregoing this vector represents in magnitude and direction the rate of bending of the curve at the point P, the bending taking place in the plane through P at right angles to this vector.*

In the case of a plane curve this vector curvature is always parallel to a fixed direction—that of the perpendicular to the plane, but in the general case the direction of the vector is continually changing. The plane through P to which it is perpendicular, or the plane of the bending at P, is the *osculating plane* of the curve at P.

To investigate the rate of rotation of the osculating plane as we pass along the curve, or, what is equivalent, the rate of rotation of the normal $UVd\rho d^2\rho$ to that plane (compare the third form of (III.)), we have by (II.),

$$\frac{dUVd\rho d^2\rho}{UVd\rho d^2\rho\,.\,Td\rho}=V\,.\,\frac{Vd\rho d^3\rho}{Vd\rho d^2\rho}\cdot\frac{1}{Td\rho}=Ud\rho\,.\,S\frac{d^3\rho}{Vd\rho d^2\rho}, \qquad\text{(IV.)}$$

since $dVd\rho d^2\rho=Vd\rho d^3\rho$. This is the *vector torsion* of the curve at P. It gives in magnitude and direction the rate of rotation of the osculating plane, and we see (what is geometrically obvious) that the osculating plane rotates about the tangent line $(Ud\rho)$.

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* The phrases *vector curvature* and *vector torsion* correspond to Hamilton's *vector of curvature* and *vector of second curvature.* We shall see what advantage results from considering an angular velocity to be a vector on the plan of this article, and the present case is quite analogous. It is easier in Quaternions to represent the primary characteristics of a curve, the curvature and the torsion, by vectors than to represent the somewhat artificial and indirect conception of an osculating circle or radius of torsion. The theory of emanant lines has been worked out by Hamilton (*Elements of Quaternions*, Art. 396).

The vector curvature and the vector torsion may be compounded into a single rate of rotation

$$\omega = \mathrm{V}\frac{\mathrm{d}\rho\mathrm{d}^2\rho}{\mathrm{T}\mathrm{d}\rho^3} + \mathrm{U}\mathrm{d}\rho\,.\,\mathrm{S}\frac{\mathrm{d}^3\rho}{\mathrm{V}\mathrm{d}\rho\mathrm{d}^2\rho}, \quad \text{(v.)}$$

which may perhaps be called the *vector twist* of the curve. This rotation produces the same effect on the tangent line and on the osculating plane as the vector curvature and the vector torsion respectively, for the former vector is at right angles to the osculating plane and the latter is parallel to the tangent line, and we do not here consider the rotation of the osculating plane in its plane or the rotation of the tangent line round itself.

If the equation of the curve is given in the form considered in Art. 48, that is if ρ is given as a function of a parameter t, the expression (v.) may be written in the form

$$\omega = \frac{\mathrm{V}\rho'\rho''}{\mathrm{T}\rho'^3} + \mathrm{U}\rho'\mathrm{S}\frac{\rho'''}{\mathrm{V}\rho'\rho''} \quad \text{(vi.)}$$

where ρ', ρ'' and ρ''' are the successive deriveds of ρ with respect to the parameter.

If the arc of the curve is taken as the independent variable, and if ρ_1, ρ_2, ρ_3, etc., denote the successive deriveds of ρ with respect to the arc, the relations (compare Art. 48, p. 63)

$$\mathrm{T}\rho_1 = 1, \quad \mathrm{S}\rho_1\rho_2 = 0, \quad \mathrm{S}\rho_1\rho_3 + \rho_2{}^2 = 0, \quad \text{etc.}, \quad \text{(vii.)}$$

found by equating to zero the successive deriveds of $\mathrm{T}\rho_1$, serve to simplify the various formulae. Thus (v.) becomes

$$\omega = \rho_1\rho_2 + \rho_1\mathrm{S}\frac{\rho_3}{\rho_1\rho_2}. \quad \text{(viii.)}$$

Ex. 1. Show how to connect the deriveds of ρ taken with respect to t and with respect to s.

$$\left[\rho' = \rho_1\frac{\mathrm{d}s}{\mathrm{d}t}, \quad \rho'' = \rho_2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + \rho_1\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}, \text{ etc.}\right]$$

Ex. 2. Show that the tangent line and the osculating plane of any curve may be written respectively in the forms,

$$\varpi = \rho + x\rho', \quad \varpi = \rho + x\rho' + y\rho'',$$

x and y being variable scalars.

Ex. 3. The tangent line and osculating plane of the twisted cubic

$$\varpi = (\phi + t)^{-1}\alpha$$

may be expressed by

$$\varpi = (\phi + x)(\phi + t)^{-2}\alpha, \quad \varpi = (\phi + x)(\phi + y)(\phi + t)^{-3}\alpha,$$

respectively, α being a constant vector and ϕ a given linear vector function.

Ex. 4. Calculate the vector ω for the helix

$$\varpi = a(i\cos t + j\sin t) + kbt,$$

i, j and k being mutually rectangular unit vectors.

Ex. 5. Find the centre of the osculating circle of a curve.

[The vector to the centre from the point on the curve has the same direction as $V\rho'\rho''T\rho'^{-3}\,.\,U\rho'$, and its tensor is the reciprocal of that of this vector.]

ART. **86**. The important relations (II.) and (IV.) of the last article enable us to reduce every affection of the curve to a function of the unit vectors

$$a = Ud\rho, \quad \gamma = UVd\rho d^2\rho, \quad \beta = UVd\rho d^2\rho Ud\rho, \quad \text{(I.)}$$

of the scalars

$$c_1 = \frac{TVd\rho d^2\rho}{Td\rho^3}, \quad a_1 = S\frac{d^3\rho}{Vd\rho d^2\rho}, \quad \text{(II.)}$$

and of the deriveds of these scalars with respect to the arc.

We notice first that a, β and γ form a mutually rectangular unit system so that $a\beta = \gamma$, $\beta\gamma = a$, $\gamma a = \beta$. The scalars a_1 and c_1 are the ordinary scalar torsion and curvature respectively, and partly for the sake of symmetry we regard them as the deriveds $\frac{da}{ds}, \frac{dc}{ds}$ of two angles a and c. The angle a is the total angle through which the osculating plane has turned about the tangent line in passing from some initial point P_0 on the curve to the point P. In like manner c is the total or integrated angle through which the tangent line has turned in the osculating plane from P_0 to P. The vector a is along the tangent, β along the *principal normal* and γ along the *binormal* to the curve.

Denoting still deriveds with respect to the arc s by suffixes, the fundamental formulae, (II.) and (IV.) of the last article, give in accordance with (I.) and (II.) of the present, the simple relations

$$\frac{a_1}{a} = c_1\gamma, \quad \frac{\gamma_1}{\gamma} = a_1 a, \quad \frac{\beta_1}{\beta} = a_1 a + c_1\gamma = \omega, \quad \text{(III.)}$$

or

$$a_1 = c_1\beta, \quad \beta_1 = a_1\gamma - c_1 a, \quad \gamma_1 = -a_1 a, \quad \text{(IV.)}$$

or simply

$$\eta_1 = V\omega\eta, \quad \text{(V.)}$$

if η stands for a, β or γ.

The formulae in a and γ are translations of the formulae of the last article. The formula in β is derived from these by aid of the relation $\beta = \gamma a$.

To express the successive deriveds, with respect to the arc, of the vector to any point on the curve in terms of a, β, γ and of the scalars a_1, c_1 and the deriveds a_2, c_2, etc., of these scalars, we have

$$\left.\begin{aligned}
&\rho_1 = a,\\
&\rho_2 = a_1 = \beta c_1,\\
&\rho_3 = a_2 = \beta_1 c_1 + \beta c_2 = \beta c_2 + (\gamma a_1 - a c_1)c_1,\\
&\rho_4 = \beta c_3 + 2(\gamma a_1 - a c_1)c_2 + (\gamma a_2 - a c_2)c_1 - \beta(a_1^2 + c_1^2)c_1\,;
\end{aligned}\right\} \text{(VI.)}$$

and in general we shall find the n^{th} derived to be of the form

$$\rho_n = aA_n + \beta B_n + \gamma C_n, \qquad \text{(VII.)}$$

where A_n, B_n and C_n are certain scalars (*not* the n^{th} deriveds of scalars A, B, C, however). We may remark that the deriveds of highest order of c_1 and a_1 occur in ρ_n in the term $\beta c_n + \gamma a_{n-1} c_1$, as we see from (VI.).

Thus, as we have asserted, every affection of the curve may be expressed in terms of a, β, γ of a_1 and c_1, and of the deriveds of these scalars. (See Appendix. *Elements*, Vol. ii.)

ART. **87**. The developables connected with the curve may all be investigated in one common way.

The vector η and the scalar e being in some way variable with a point on a curve, a plane of any developable connected with the curve is expressible by an equation of the form

$$S(\varpi - \rho)\eta = e, \qquad \text{(I.)}$$

ϖ being the variable vector to a point in the plane, and ρ being the vector to the point P on the curve to which the plane corresponds. The equation of a successive plane is of the form

$$S(\varpi - \rho)\eta - e + ds \,.\, \frac{\partial}{\partial s}(S(\varpi - \rho)\eta - e) = 0, \qquad \text{(II.)}$$

e, η and ρ being regarded as functions of the arc s, but ϖ being independent of s. Thus two successive planes intersect in the line of intersection of the first plane and of the plane determined by equating to zero its derived with respect to s. The intersection of the plane (I.) and its consecutive is accordingly the line common to (I.) and to the plane

$$S(\varpi - \rho)\eta_1 = Sa\eta + e_1, \qquad \text{(III.)}$$

η_1 and e_1 being the first deriveds of η and e.

This line of the developable is also given by the vector equation (Art. 35 (I.), p. 39),

$$\varpi = \rho + \frac{\eta_1 e - \eta(Sa\eta + e_1) + t}{V\eta\eta_1}, \qquad \text{(IV.)}$$

where t is a variable parameter.

In the same way, equating to zero the second derived of (I.) with respect to s,

$$S(\varpi - \rho)\eta_2 = 2Sa\eta_1 + S\beta\eta \,.\, c_1 + e_2, \qquad \text{(V.)}$$

and combining this with (III.) and (I.), we have the point of intersection of three successive planes of the developable,

$$\varpi = \rho + \frac{V\eta_1\eta_2 e + V\eta_2\eta(Sa\eta + e_1) + V\eta\eta_1(2Sa\eta_1 + S\beta\eta \,.\, c_1 + e_2)}{S\eta\eta_1\eta_2}. \qquad \text{(VI.)}$$

This point is on the cuspidal edge of the developable, and it corresponds to the point P on the curve. More generally if in (VI.) we allow the arc to vary, we have the equation of the cuspidal edge of the developable.

In particular, the *polar* developable corresponds to $\eta=\alpha$, $e=0$; while $\eta=\beta$, $e=0$ gives the *rectifying* developable; and $\eta=\gamma$, $e=0$ is the tangent line developable. It is shorter in many cases to treat the developables *ab initio* rather than to substitute in the general formulae (IV.) and (VI.).

Ex. 1. The vectors from a point on the curve to the centres of the osculating circle and sphere are respectively

$$\frac{\beta}{c_1} \quad \text{and} \quad \frac{\beta}{c_1}+\gamma\frac{\mathrm{d}}{\mathrm{d}a}\cdot\frac{1}{c_1}.$$

[These expressions follow from consideration of the polar developable. Or the first is geometrically obvious, and it is also evident that the centre of spherical curvature lies on the polar line, $\varpi=\rho+\frac{\beta}{c_1}+x\gamma$, which is by geometry the locus of points equidistant from three consecutive points on the curve. To determine x we may express that ϖ is the vector to a point which is momentarily stationary as we pass along the curve. Thus

$$\frac{\mathrm{d}\varpi}{\mathrm{d}s}=0=\alpha+\frac{\gamma a_1-\alpha c_1}{c_1}+\beta\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{c_1}\right)-x\beta a_1+\frac{\delta x}{\mathrm{d}s}\cdot\gamma, \text{ and therefore } x=\frac{1}{a_1}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{c_1}\right).$$

We must remember that x is not here a function of s. δx is some small scalar. See the next example.]

Ex. 2. For a spherical curve

$$\frac{1}{c_1}+\frac{\mathrm{d}^2}{\mathrm{d}a^2}\left(\frac{1}{c_1}\right)=0.$$

[In this case we can determine x so that the vector in the last example terminates at a fixed point in the centre of the sphere containing the curve, and now $\delta x:\mathrm{d}s$ is the derived of x with respect to s, so that

$$-\frac{a_1}{c_1}=\frac{\delta x}{\mathrm{d}s}=\frac{\mathrm{d}x}{\mathrm{d}s}=\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{a_1}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{c_1}\right)\right).$$

The method here employed is often useful. The condition may also be found by expressing that the vector to the centre of spherical curvature terminates at a fixed point. The condition is momentarily true (not an identity) if five consecutive points lie on a sphere.]

Ex. 3. Prove that the rectifying line is $\mathrm{V}(\varpi-\rho)\omega=0$, and that the cuspidal edge of the rectifying developable is $\varpi=\rho-\frac{\omega}{c_1}:\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{a_1}{c_1}\right)$.

[The rectifying plane $\mathrm{S}(\varpi-\rho)\beta=0$ through the tangent line and at right angles to the osculating plane, generates this developable.]

Ex. 4. The curve is a geodesic on the rectifying developable.

[Prove that the angles of the quaternions

$$(\omega+\mathrm{d}\omega):\alpha \quad \text{and} \quad (\omega+\mathrm{d}\omega):(\alpha+\mathrm{d}\alpha)$$

are equal to the second order of small quantities, and hence show that when the developable is flattened out the curve becomes a right line, so that it is a line of shortest distance (or a geodesic) on the developable.]

Ex. 5. If the ratio of curvature (scalar) to torsion is constant, the curve is a geodesic on a cylinder.

[If $T\omega \,.\, \cos H = a_1$, $T\omega \sin H = c_1$, the angle H is here constant, and equations (III.), Art. 86, give $d(\alpha \cos H + \gamma \sin H) = 0$, or on integration $U\omega = k$, a constant vector. The rectifying developable is therefore a cylinder.]

Ex. 6. Show how to determine the curves for which the ratio of curvature to torsion is constant.

[By the last example we have $\alpha_1 = \gamma\alpha \sin H \,.\, T\omega = Vk\alpha \,.\, T\omega$. If $dt = T\omega \,.\, ds$, we have, on changing the variable from s to t, $\alpha' = Vk\alpha$, and on differentiating,

$$\alpha'' = Vk\alpha' = kVk\alpha = -\alpha - kSk\alpha = -\alpha + k\cos H;$$

or

$$\frac{d^2}{dt^2}(\alpha - k\cos H) + (\alpha - k\cos H) = 0.$$

The integral of this equation is $\alpha - k\cos H = \lambda \cos t + \mu \sin t$, and as we must have $Sk\alpha = -\cos H$ and $T\alpha = 1$, it appears that λ and μ must be perpendicular to one another and to k, and that their tensors must be equal to $\sin H$. Thus

$$\alpha = k\cos H + \sin H(i\cos t + j\sin t),$$

and on integrating again

$$\varpi = \int \alpha ds = \rho_0 + ks\cos H + \sin H \,.\, \int (i\cos t + j\sin t)ds,$$

where ρ_0 is a vector constant of integration.]

Ex. 7. Find the conditions that the unit vectors (α, β, γ) of one curve may remain constantly inclined to those $(\alpha', \beta', \gamma')$ at corresponding points of another.

[We must have $\omega ds = \omega' ds'$, or $\alpha da + \gamma dc = \alpha' da' + \gamma' dc'$. Hence either $\beta' \parallel \beta$ or else $da : dc = -S\gamma\beta' : S\alpha\beta' = const.$ In the second case both curves are geodesics on cylinders. In the first, if α' makes the angle u with α, γ' makes the same angle with γ (the four vectors being coplanar), and $H = u + H'$. In other words,

$$da = \cos u \,.\, da' - \sin u \,.\, dc', \quad dc = \sin u \,.\, da' + \cos u \,.\, dc'.]$$

Ex. 8. Find the unit vectors for the locus of centres of spherical curvature, and show that they remain constantly inclined to those of the given curve.

Ex. 9. The vectors ρ and ρ' are drawn from a centre of reciprocation to a point on a curve and to the corresponding point on the cuspidal edge of the developable into which the curve reciprocates, prove that

$$\rho'\gamma S\gamma\rho = \rho\gamma' S\gamma'\rho' = K^2,$$

where K is the radius of reciprocation and where γ and γ' are unit vectors normal to the osculating planes at ρ and ρ'.

(a) Compare the curvatures and torsions of the two curves.

Ex. 10. Compare the unit vectors for a curve and its inverse.

(ii) *Ruled Surfaces.*

ART. **88.** Having showed in the last article how to determine the surfaces generated by planes connected with the curve, we shall now consider the surfaces generated by the emanant line (compare Art. 85, p 131)

$$V(\varpi - \rho)\eta = 0. \quad \text{..............................(I.)}$$

Reverting to p. 40, Art. 36 (V.), the shortest vector QQ' from the line to any other line $V(\varpi-\rho')\eta'=0$ is

$$QQ'=V\eta\eta' S\frac{\rho'-\rho}{V\eta\eta'}, \text{ and } OQ=\rho+\eta S\frac{(\rho'-\rho)\eta'}{V\eta\eta'}. \quad \text{........(II.)}$$

Putting in these $\rho'=\rho+d\rho$, $\eta'=\eta+d\eta$ and proceeding to the limit having divided $Q'Q$ by $T(\rho'-\rho)$, we find

$$\frac{QQ'}{Td\rho}=V\eta d\eta S\frac{Ud\rho}{V\eta d\eta}=\iota S\frac{Ud\rho}{\iota}=\frac{dU\eta}{U\eta}S\frac{Ud\rho U\eta}{dU\eta}=p\iota, \quad \text{....(III.)}$$

by Art. 85 (II.); and neglecting a vanishing term in the expression for OQ,

$$OQ=\rho+\eta S\frac{d\rho\eta}{V\eta d\eta}=\rho-U\eta S\frac{d\rho}{dU\eta}=\rho-U\eta S\frac{Ud\rho}{\iota U\eta}, \quad \text{......(IV.)}$$

the various transformations being easy consequences of the formula just cited, and p being a scalar defined by

$$p=S\frac{Ud\rho}{\iota}=S\frac{d\rho}{Vd\eta\eta^{-1}}=S\frac{d\rho U\eta}{dU\eta}. \quad \text{..................(V.)}$$

The vector $p\iota$ represents the *rate of translation* of the emanant line as it passes through successive positions, this vector being the ratio of the shortest distance between consecutive positions to the arc ds of the curve. In other words, the emanant may be supposed to pass from one position to a consecutive in virtue of a rotation ιds about the shortest distance QQ' coupled with a translation $QQ'=p\iota ds$ along that shortest distance. Or again p is the ratio of the shortest distance to the angle between the consecutive lines. The quantity p is usually called the *parameter of distribution* of the ruled surface, though the theory of screws would offer the more suggestive term *pitch*, because the transference of the generator from one position to the consecutive is in the language of the theory of screws effected by a twist about the screw coaxial with the shortest distance and of pitch p. The point Q, the extremity of the vector (IV.), is the point of closest approach of successive generators; and as s varies Q describes the *line of striction* of the ruled surface. For a developable, this coincides with the cuspidal edge, and p vanishes.

Ex. 1. Prove that the line of striction and the parameter of distribution of the surface generated by the principal normals of a curve are

$$\varpi=\rho+\frac{\beta c_1}{a_1^2+c_1^2}, \quad p=\frac{a_1}{a_1^2+c_1^2}.$$

Ex. 2. The tangent to the line of striction of this surface is parallel to

$$\frac{\omega a_1}{a_1^2+c_1^2}+\beta\frac{d}{ds}\left(\frac{c_1}{a_1^2+c_1^2}\right),$$

and the shortest distance between consecutive generators is parallel to ω.

Ex. 3. If $\eta = a\cos l + \beta \sin l \cos m + \gamma \sin l \sin m$,
prove that the condition that the emanant line η should generate a developable is

$$\sin l \,.\, \mathrm{d}(a+m) - \cos l \sin m \mathrm{d}c = 0 \text{ or } \sin l = 0.$$

[By (v.) if $p=0$, $\mathrm{S}a\eta \mathrm{d}\eta = 0$.]

Ex. 4. Prove that no line except a in the plane of a and β can generate a developable; that the only developables generated by lines in the plane of a and γ are the tangent-line and the rectifying developables; and that any line whatever in the plane of β and γ is capable of generating a developable.

[For the plane of a and β, $l=0$ or $m=0$, and $m=0$ is impossible if a varies. For the plane of a and γ, $l=0$ or $m=\frac{\pi}{2}$. If $m=\frac{\pi}{2}$, we find $\eta = \mathrm{U}\omega$ since $\sin l \,.\, \mathrm{d}a = \cos l \,.\, \mathrm{d}c$. If $l=\frac{\pi}{2}$, we have a series of developables

$$\varpi = \rho + t(\beta \cos(a-a_0) - \gamma \sin(a-a_0))\,;$$

and their cuspidal edges are

$$\varpi = \rho + \frac{\beta}{c_1} - \frac{\gamma}{c_1}\tan(a-a_0),$$

a_0 being an arbitrary constant.]

Ex. 5. Prove that the curves

$$\varpi = \rho + \frac{\beta}{c_1} - \frac{\gamma}{c_1}\tan(a-a_0)$$

are the evolutes of the curve $\varpi=\rho$, and that they lie in the polar developable.

Ex. 6. If the emanant is perpendicular to the tangent, prove that

$$\mathrm{OQ} = \rho + \frac{\eta c_1 \cos m}{c_1^2\cos^2 m + (a_1+m_1)^2}, \quad p = \frac{a_1+m_1}{c_1^2\cos^2 m + (a_1+m_1)^2},$$

where $\eta = \beta\cos m + \gamma \sin m$.

ART. **89.** The normal to the ruled surface

$$\varpi = \rho + u\eta \qquad \text{(I.)}$$

at any point ϖ is parallel to

$$\nu = \mathrm{V}\eta(\mathrm{d}\rho + u\mathrm{d}\eta), \qquad \text{(II.)}$$

this vector being perpendicular to every tangential vector

$$\mathrm{d}\varpi = \mathrm{d}\rho + u\mathrm{d}\eta + \eta \mathrm{d}u. \qquad \text{(III.)}$$

The tangent plane is

$$\mathrm{S}(\varpi - \rho)\mathrm{V}\eta(\mathrm{d}\rho + u\mathrm{d}\eta) = 0, \qquad \text{(IV.)}$$

and as it generally involves u, it varies from point to point along the generator. Moreover, since it involves u linearly, the anharmonic of four tangent planes is equal to the anharmonic of the four corresponding normal vectors (II.), or of the four corresponding points of contact (I.), (Art. 37, p. 41).

Expressing that the tangent planes at two points u and u' on the same generator are perpendicular, we have a relation

$$\mathrm{S}\nu\nu' = 0, \text{ or } \mathrm{SV}\eta(\mathrm{d}\rho + u\mathrm{d}\eta)\mathrm{V}\eta(\mathrm{d}\rho + u'\mathrm{d}\eta) = 0, \qquad \text{(V.)}$$

which determines an involution between the corresponding points u and u'. This may be thrown into the form

$$\left(u+\mathrm{S}\frac{\mathrm{V}\eta\mathrm{d}\rho}{\mathrm{V}\eta\mathrm{d}\eta}\right)\left(u'+\mathrm{S}\frac{\mathrm{V}\eta\mathrm{d}\rho}{\mathrm{V}\eta\mathrm{d}\eta}\right)=-\mathrm{T}\left(\mathrm{V}\frac{\mathrm{V}\eta\mathrm{d}\rho}{\mathrm{V}\eta\mathrm{d}\eta}\right)^2, \qquad \text{(VI.)}$$

because $(\mathrm{S}\lambda\mu^{-1})^2-\mathrm{T}(\lambda\mu^{-1})^2=-\mathrm{T}(\mathrm{V}\lambda\mu^{-1})^2$. Comparing with equation (IV.) of the last article, it appears that the point Q in which the generator meets the line of striction is the centre of the involution, and that the foci are imaginary. If C and C′ are the two points u and u', it is not difficult to see that this equation (VI.) is equivalent to

$$\mathrm{QC}\,.\,\mathrm{QC}'=+p^2, \qquad \text{(VII.)}$$

QC and QC′ being vectors, and because their product is positive, they must be oppositely directed. That the quantity on the right in (VI.) reduces to $\mathrm{T}\eta^{-2}p^2$ follows most easily by taking the arc as the independent variable, and then

$$\begin{aligned}\mathrm{V}\,.\,\mathrm{V}\eta\rho_1(\mathrm{V}\eta\eta_1)^{-1}&=\eta\mathrm{S}\rho_1(\mathrm{V}\eta\eta_1)^{-1}=\eta\mathrm{T}\eta^{-2}\mathrm{S}\rho_1(\mathrm{V}\eta_1\eta^{-1})^{-1}\\&=-\eta^{-1}\mathrm{SUd}\rho\,.\,\iota^{-1}\end{aligned}$$

by (V.) of the last article.

Ex. 1. If the tangent planes of a ruled surface touch the surface all along the generators, the surface must be a developable or a cylinder.

[The direction of the normal must be independent of u. This requires $\mathrm{d}\eta \parallel \eta$, that is, $\mathrm{dU}\eta=0$, or else $\mathrm{d}\rho \parallel \eta$, or the line is a tangent to the curve $\varpi=\rho$.]

Ex. 2. If for any point $p=0$ the tangent plane touches all along the generator.

[A generator of this kind is said to be *torsal*. A ruled surface has in general a definite number of torsal generators.]

Ex. 3. The point Q being on the line of striction, prove that the tangent of the angle between the tangent planes at Q and at any point C on the same generator is

$$\tan \mathrm{A}=\frac{\mathrm{TCQ}}{p}.$$

Ex. 4. Prove that the vector velocities of the points C and C′ are at right angles, and compare their magnitudes.

[The vector velocity of C is $\iota(\mathrm{QC}+p)$. See Art. 88.]

Ex. 5. Prove that the vector to a point on the line of striction of the quadric $\mathrm{S}\rho\phi\rho+1=0$, and the corresponding parameter of distribution are respectively

$$\rho=\pm\sqrt{-\frac{1}{m}}\mathrm{V}\frac{\phi^2\eta}{\phi\eta},\quad p=\pm\sqrt{-\frac{1}{m}}\mathrm{S}\frac{\psi\eta}{\phi\eta},$$

where $\mathrm{S}\eta\phi\eta=0$.

[See Art. 88. To reduce we may take η to be a unit vector so that $\mathrm{S}\eta\eta'=0$, $\mathrm{S}\eta'\phi\eta=0$ as well as $\mathrm{S}\eta\phi\eta=0$.]

(iii) *Curvature of Surfaces.*

ART. **90**. Projecting a curve on any plane, normal to the fixed vector k, the curvature of the projection is (Art. 85 (III.), p. 132)

$$\frac{\mathrm{d}\,.\,\mathrm{Ud}\,.\,k^{-1}\mathrm{V}k\rho}{\mathrm{d}\,.\,k^{-1}\mathrm{V}k\rho}=\frac{\mathrm{V}\,.\,k^{-1}\mathrm{V}k\mathrm{d}\rho\,.\,k^{-1}\mathrm{V}k\mathrm{d}^2\rho}{\mathrm{T}(k^{-1}\mathrm{V}k\mathrm{d}\rho)^3}=\frac{k^{-1}\mathrm{S}k\mathrm{d}\rho\mathrm{d}^2\rho}{\mathrm{T}(\mathrm{V}k\mathrm{d}\rho)^3}$$

$$=k^{-1}\mathrm{S}k\frac{\mathrm{dUd}\rho}{\mathrm{d}\rho}\cdot\frac{\mathrm{Td}\rho^3}{\mathrm{T}(\mathrm{V}k\mathrm{d}\rho)^3},\qquad\text{(I.)}$$

or *the curvature of the projection is the projection of the curvature into the cube of the cosecant of the angle between the tangent to the curve and the normal to the plane of projection.*

If the plane of projection is parallel to the tangent, the projection of the curvature is the curvature of the projection.

Resolving the vector curvature of a curve traced on a surface into its components perpendicular to and along the normal ν, we have

$$\frac{\mathrm{dUd}\rho}{\mathrm{d}\rho}=\nu^{-1}\mathrm{V}\nu\frac{\mathrm{dUd}\rho}{\mathrm{d}\rho}+\nu^{-1}\mathrm{S}\nu\frac{\mathrm{dUd}\rho}{\mathrm{d}\rho},\qquad\text{(II.)}$$

and since $\mathrm{S}\nu\mathrm{d}\rho=0$, the first component is, by what we have just proved, the curvature of the projection of the curve on the normal plane ($\perp\nu\mathrm{d}\rho$) to the surface through the tangent line, and the second is the curvature of the projection on the tangent plane.

Remembering that $\mathrm{S}\nu\mathrm{d}\rho=0$, and that its derived is also zero, or $\mathrm{S}\nu\mathrm{d}^2\rho=-\mathrm{Sd}\nu\mathrm{d}\rho$, the first component admits of the transformations

$$\nu^{-1}\mathrm{V}\nu\frac{\mathrm{dUd}\rho}{\mathrm{d}\rho}=\nu^{-1}(\mathrm{d}\rho)^{-1}\mathrm{S}\nu\mathrm{dUd}\rho=\frac{\mathrm{S}\nu\mathrm{d}^2\rho}{\mathrm{d}\rho\,.\,\nu\,.\,\mathrm{Td}\rho}=-\frac{\mathrm{Sd}\nu\mathrm{d}\rho}{\mathrm{d}\rho\,.\,\nu\,.\,\mathrm{Td}\rho}.\qquad\text{(III.)}$$

The last of these shows that the component is the same for all curves traced on the surface, provided they have a common tangent line $\mathrm{d}\rho$, $\mathrm{d}\nu$ being a linear function of $\mathrm{d}\rho$; and thus in particular it is the curvature of the normal section of the surface through $\mathrm{d}\rho$. This is Meusnier's theorem—the magnitude of the curvature of the normal section is that of the oblique section into the cosine of the angle between their planes.

The second component is, as we have already shown, the curvature of the projection of the curve on the tangent plane, or it is the rate of bending of the curve round the normal (or in the tangent plane). It vanishes for a geodesic—the straightest curve on the surface between a pair of points—for such a curve can have no component of bending in the tangent plane; and it is called the *geodesic* curvature of the curve. The differential equation of a geodesic is therefore

$$\mathrm{S}\nu\mathrm{d}\rho\mathrm{dUd}\rho=0,\ \text{ or }\ \mathrm{S}\nu\mathrm{d}\rho\mathrm{d}^2\rho=0.\qquad\text{(IV.)}$$

The normals to the surface along the curve trace out a ruled surface, and by Art. 88 the equation of the line of striction and the value of the parameter of distribution are

$$OQ=\rho+\nu S\frac{d\rho\nu}{V\nu d\nu},\quad p=S\frac{d\rho}{Vd\nu\nu^{-1}}. \qquad \text{(v.)}$$

The tangent planes along the curve generate a developable. This and its cuspidal edge are respectively represented by

$$\varpi=\rho+uV\nu d\nu,\quad \varpi=\rho+\frac{V\nu d\nu Sd\rho d\nu}{S\nu d\nu d^2\nu}. \qquad \text{(vi.)}$$

ART. **91**. If $f\rho$ is any scalar function of ρ, and if we write

$$df\rho=nS\nu d\rho,\quad d\nu=\phi d\rho, \qquad \text{(i.)}$$

the function ϕ is self-conjugate when n is independent of ρ or when it is a function* of $f\rho$.

Let $d\rho$ and $d'\rho$ be any two independent differentials of ρ so that

$$d'd\rho=dd'\rho,\quad d'df\rho=dd'f\rho. \qquad \text{(ii.)}$$

We find on expansion by (i.) if $dn=S\sigma d\rho$,

$$d'df\rho=nS\phi d'\rho d\rho+nS\nu d'd\rho+S\sigma d'\rho S\nu d\rho,$$

$$dd'f\rho=nS\phi d\rho d'\rho+nS\nu dd'\rho+S\sigma d\rho S\nu d'\rho;$$

and by (ii.) these expressions give

$$Sd\rho(n\phi d'\rho+\nu S\sigma d'\rho)=Sd'\rho(n\phi d\rho+\nu S\sigma d\rho). \qquad \text{(iii.)}$$

The function $n\phi\varpi+\nu S\sigma\varpi$ is therefore self-conjugate; and if n is constant so that σ is zero, or if it is a function of $f\rho$ so that $\sigma \parallel \nu$, the function ϕ is self-conjugate likewise. We also observe that if ϵ is the spin-vector of ϕ,

$$2n\epsilon+V\nu\sigma=0 \quad \text{and} \quad S\nu\epsilon=0. \qquad \text{(iv.)}$$

This scalar condition is in fact the condition that $S\nu d\rho=0$ should lead to an integral $f\rho=\text{const.}$

If the equation of a surface is given in the form $f\rho=\text{const.}$, the differential vanishes if $d\rho$ is any tangential vector at the extremity of the vector ρ, and the vector ν is parallel to the normal.

ART. **92**. In applying the results of the last article to the study of surfaces, we shall leave $T\nu$ arbitrary, and shall write $\phi=\phi_0+V\epsilon$. The spin-vector ϵ disappears automatically from $Sd\rho d\nu=Sd\rho\phi d\rho=Sd\rho\phi_0 d\rho$, whatever vector $d\rho$ may be, and it also disappears from $V\nu d\nu=V\nu(\phi_0+V\epsilon)d\rho$, because in this case $S\nu d\rho=0$ and also $S\nu\epsilon=0$ by (iv.) of the last article, so that $V\nu V\epsilon d\rho=0$. Thus we have

$$d\nu=\phi d\rho=(\phi_0+V\epsilon)d\rho,\quad Sd\rho d\nu=Sd\rho\phi_0 d\rho,\quad V\nu d\nu=V\nu\phi_0 d\rho. \qquad \text{(i.)}$$

* This is included in a more general theorem (Art. 60, p. 80).

Writing C for the magnitude* of the curvature of the normal section parallel to $d\rho$,

$$C = -\frac{Sd\rho\phi_0 d\rho}{T\nu Td\rho^2}, \quad S\nu d\rho = 0; \quad\text{(II.)}$$

and it follows at once by Art. 73, p. 107, that C is the inverse square of the radius of the conic

$$S\varpi\phi_0\varpi = -T\nu, \quad S\nu\varpi = 0, \quad\text{(III.)}$$

which is parallel to $d\rho$. It is also evident from (I.) that $V\nu d\nu$ is parallel to the radius of this conic conjugate to $d\rho$.

Remembering that the function ϕ_0 is independent of $d\rho$, although it involves ρ in its constitution, we may for any point on the surface regard ϕ_0 as constant, and we may apply the formulae of Art. 75 to calculate the directions of the principal axes of the conic (III.). The inverse squares of the principal radii of the conic are the principal curvatures (C_1 and C_2) of the surface, and are the roots of the quadratic

$$S\nu(\phi_0 - CT\nu)^{-1}\nu = 0, \quad\text{or}\quad C^2T\nu^2 - CS\nu^{-1}\chi_0\nu + S\nu^{-1}\psi_0\nu = 0; \quad\text{(IV.)}$$

and unit vectors (τ_1 and τ_2) along the principal axes are determined by

$$\tau_1 \parallel (\phi_0 - C_1T\nu)^{-1}\nu, \quad \tau_2 \parallel (\phi_0 - C_2T\nu)^{-1}\nu. \quad\text{(V.)}$$

The three vectors τ_1, τ_2 and $U\nu$ form a mutually rectangular unit vector system, and we suppose the directions chosen so that $\tau_1\tau_2 = U\nu$.

Writing also

$$Ud\rho = \tau_1 \cos l + \tau_2 \sin l, \quad\text{(VI.)}$$

the expression for the curvature (II.) of the normal section reduces to

$$C = C_1 \cos^2 l + C_2 \sin^2 l; \quad\text{(VII.)}$$

by (I.) we also have

$$V\nu d\nu = (\tau_2 C_1 \cos l - \tau_1 C_2 \sin l)T\nu^2 Td\rho, \quad\text{(VIII.)}$$

since $$V\tau_1\tau_2\phi_0 Ud\rho = \tau_1 S\tau_2\phi_0 Ud\rho - \tau_2 S\tau_1\phi_0 Ud\rho$$
$$= \tau_1 S\tau_2\phi_0\tau_2 \sin l - \tau_2 S\tau_1\phi_0\tau_1 \cos l;$$

and the vector OQ to the point of closest approach of consecutive normals along $d\rho$ and the scalar p (Art. 90, (V.)), assume the forms

$$OQ = \rho - U\nu \cdot \frac{C_1 \cos^2 l + C_2 \sin^2 l}{C_1^2 \cos^2 l + C_2^2 \sin^2 l}, \quad p = \frac{(C_1 - C_2)\sin l \cos l}{C_1^2 \cos^2 l + C_2^2 \sin^2 l}. \quad\text{(IX.)}$$

* It is not hard to see by considering the sense of rotation that if we suppose C to be positive for a surface like an ellipsoid, the sign selected in (II.) requires ν to be drawn on the *convex* side. Of course there is no ambiguity about the *vector* curvature.

The quadric

$$S(\varpi-\rho)\phi_0(\varpi-\rho)+2S\nu(\varpi-\rho)=0, \qquad \text{(X.)}$$

in which ϖ is variable and ρ constant, has complete contact of the second order with the surface. We have in fact at the point $\varpi=\rho$, $0=Sd\varpi\phi_0 d\varpi+S\nu d^2\varpi$, where $d\varpi$ and $d^2\varpi$ are differentials of ϖ as terminating on the quadric, and this is also true for differentials of the vector to a point terminating on the surface. The equation of the quadric may also be written in the form

$$S(\varpi-\rho+\phi_0{}^{-1}\nu)\phi_0(\varpi-\rho+\phi_0{}^{-1}\nu)=S\nu\phi_0{}^{-1}\nu, \qquad \text{(XI.)}$$

and it is not difficult to prove that the principal curvatures are the parameters of the confocal quadrics

$$S(\varpi-\rho+\phi_0{}^{-1}\nu).(\phi_0{}^{-1}-C_1{}^{-1}T\nu^{-1})^{-1}(\varpi-\rho+\phi_0{}^{-1}\nu)=S\nu\phi_0{}^{-1}\nu \qquad \text{(XII.)}$$

which pass through the extremity of ρ. The subject will be resumed in Art. 156, p. 295.

ART. **93**. The equation of the normal to a surface at the point ρ being

$$\varpi=\rho-x\nu, \qquad \text{(I.)}$$

to find the condition that two successive normals should intersect, we express that the extremity of ϖ is momentarily stationary and we have

$$d\varpi=0=d\rho-x d\nu-\nu dx=d\rho-x\phi d\rho-\nu dx, \qquad \text{(II.)}$$

where dx is some small scalar if $d\rho$ is small (see Art. 87, Ex. 1). The condition of intersection is therefore

$$Sd\rho\nu d\nu=0, \qquad \text{(III.)}$$

and this is the differential equation of the lines of curvature.

Moreover we have from (II.)

$$d\rho \parallel (1-x\phi)^{-1}\nu, \quad \text{where} \quad S\nu(1-x\phi)^{-1}\nu=0, \qquad \text{(IV.)}$$

because $S\nu d\rho=0$, and from these equations we can find the directions of the lines of curvature and the principal curvatures $C_1=x_1{}^{-1}T\nu^{-1}$, $C_2=x_2{}^{-1}T\nu^{-1}$ if x_1 and x_2 are the roots of the quadratic.

More directly, we have for the vectors to the centres of curvature,

$$\varpi_1=\rho-C_1{}^{-1}U\nu, \quad \varpi_2=\rho-C_2{}^{-1}U\nu, \qquad \text{(V.)}$$

and if $d_1\rho$ and $d_2\rho$ are tangential to these lines,

$$d_1\rho=C_1{}^{-1}d_1U\nu, \quad d_2\rho=C_2{}^{-1}d_2U\nu; \qquad \text{(VI.)}$$

and the *measure of curvature*, or the product of the principal curvatures, is

$$C_1C_2=\frac{d_1U\nu d_2U\nu}{d_1\rho d_2\rho}=\frac{VdU\nu d'U\nu}{Vd\rho d'\rho}, \qquad \text{(VII.)}$$

if $\mathrm{d}\rho$ and $\mathrm{d}'\rho$ are arbitrary tangential vectors, as we may prove by supposing ρ and $\mathrm{U}\nu$ expressed in terms of two parameters. The interpretation of this remarkable expression is that the small area determined on a unit sphere by lines drawn through its centre parallel to the normals round any small contour on the surface, bears to the area of the small contour a ratio equal to the product of the principal curvatures.

If we suppose the vector to a point on the surface to be a function of two parameters t and u, and if we use upper accents to denote differentiation with respect to t and lower accents for differentiation with respect to u, we have

$$\mathrm{d}\rho = \rho'\mathrm{d}t + \rho_{,}\mathrm{d}u,$$

and

$$\mathrm{Td}\rho^2 = e\mathrm{d}t^2 + 2f\mathrm{d}t\mathrm{d}u + g\mathrm{d}u^2,$$

if

$$e = -\rho'^2, \quad f = -\mathrm{S}\rho'\rho_{,}, \quad g = -\rho_{,}^2. \qquad \text{(VIII.)}$$

Writing also $\nu = \mathrm{V}\rho'\rho_{,}$, equation (II.) becomes

$$\rho'\mathrm{d}t + \rho_{,}\mathrm{d}u - x(\nu'\mathrm{d}t + \nu_{,}\mathrm{d}u) - \nu\mathrm{d}x = 0, \qquad \text{(IX.)}$$

and according as we eliminate x and $\mathrm{d}x$ or $\mathrm{d}t$, $\mathrm{d}u$ and $\mathrm{d}x$ we find the differential equation of the lines of curvature

$$\mathrm{d}t^2\mathrm{S}\rho'\nu'\nu - \mathrm{d}t\mathrm{d}u\mathrm{S}(\rho'\nu_{,} + \rho_{,}\nu')\nu + \mathrm{d}u^2\mathrm{S}\rho_{,}\nu_{,}\nu = 0, \qquad \text{(X.)}$$

or the equation of the principal curvatures $(C = x^{-1}\mathrm{T}\nu^{-1})$

$$C^2\mathrm{T}\nu^4 + C\mathrm{T}\nu\mathrm{S}(\rho'\nu_{,} + \nu'\rho_{,})\nu - \mathrm{S}\nu'\nu_{,}\nu = 0. \qquad \text{(XI.)}$$

It is not difficult to see that we obtain for the measure of curvature the expressions

$$\begin{aligned} C_1C_2 \,.\, \mathrm{T}\nu^4 &= \mathrm{S}\nu\rho''\mathrm{S}\nu\rho_{,,} - (\mathrm{S}\nu\rho_{,}')^2 \\ &= \mathrm{SV}\nu\rho''\mathrm{V}\nu\rho_{,,} - (\mathrm{V}\nu\rho_{,}')^2 + \nu^2(\mathrm{S}\rho''\rho_{,,} - \rho_{,}'^2); \qquad \text{(XII.)} \end{aligned}$$

and that in terms of the deriveds of e, f and g,

$$2\mathrm{V}\nu\rho'' = (e_{,} - 2f')\rho' + e'\rho_{,}, \quad 2\mathrm{V}\nu\rho_{,}' = -g'\rho' + e_{,}\rho_{,},$$

$$2\mathrm{V}\nu\rho_{,,} = -g_{,}\rho' + (2f_{,} - g')\rho_{,}, \quad 2(\mathrm{S}\rho''\rho_{,,} - \rho_{,}'^2) = e_{,,} - 2f_{,}' + g'',$$

$$\nu^2 = f^2 - eg; \qquad \text{(XIII.)}$$

and hence it follows that the measure of curvature is an explicit function of the quantities e, f and g and of their deriveds, so that the measure of curvature depends only on the expression (VIII.) of the square of a linear element. If then the surface undergoes any transformation in which the lengths of linear elements remain unchanged, the measure of curvature preserves a constant value.

ART. **94.** The following kinematical method is often useful in investigating the geometry of a surface. Suppose the vector ρ to a point on the surface to be given in terms of two parameters, u and v, and let a unit vector α be drawn at the extremity of the vector ρ tangent to the curve *u variable*; let γ be the unit vector along the normal at the same point and let $\beta=\gamma\alpha$ be at right angles to both—a tangential vector. These three variable vectors may be supposed connected with three fixed unit vectors i, j, k by the relations

$$qiq^{-1}=\alpha,\quad qjq^{-1}=\beta,\quad qkq^{-1}=\gamma\,; \quad\text{(I.)}$$

so that the conical rotation represented by q would bring the vectors i, j, k into parallelism with α, β, γ. These relations being supposed to hold for all points on the surface, it follows that q must be a function of u and v. It will be proved in Art. 106, p. 173, that if ξ is any vector function of u and v, its differential is expressible in the form,

$$\mathrm{d}\xi=\mathrm{V}(\omega'\mathrm{d}u+\omega_{,}\mathrm{d}v)\xi+\mathrm{d}(\xi), \quad\text{(II.)}$$

where* $\omega'\mathrm{d}u+\omega_{,}\mathrm{d}v=2\mathrm{V}\mathrm{d}qq^{-1}$ and $\mathrm{d}(\xi)=\alpha\mathrm{d}x+\beta\mathrm{d}y+\gamma\mathrm{d}z$ if $\xi=\alpha x+\beta y+\gamma z$, while of course $\mathrm{d}\xi$ involves differentials of α, β and γ.

We shall write in terms of α, β, γ,

$$\omega'=\alpha a'+\beta b'+\gamma c',\quad \omega_{,}=\alpha a_{,}+\beta b_{,}+\gamma c_{,}, \quad\text{(III.)}$$

so that equation (V.), Art. 106, is equivalent to

$$\frac{\partial a'}{\partial v}-\frac{\partial a_{,}}{\partial u}=b'c_{,}-b_{,}c'\,;\quad \frac{\partial b'}{\partial v}-\frac{\partial b_{,}}{\partial u}=c'a_{,}-c_{,}a'\,;\quad \frac{\partial c'}{\partial v}-\frac{\partial c_{,}}{\partial u}=a'b_{,}-a_{,}b'\,; \quad\text{(IV.)}$$

these being the results of equating coefficients of α, β, γ in the equation cited:

$$\frac{\partial(\omega')}{\partial v}-\frac{\partial(\omega_{,})}{\partial u}=\mathrm{V}\omega'\omega_{,}.$$

It will be sufficient for us to confine our attention to the case in which the curves u and v cut at right angles, so that β is tangent to v variable, since α is tangent to u variable. There is, however, no difficulty in taking the general case. We have then for the orthogonal curves,

$$\mathrm{d}\rho=A\alpha\mathrm{d}u+B\beta\mathrm{d}v \text{ and } \mathrm{T}\mathrm{d}\rho^2=A^2\mathrm{d}u^2+B^2\mathrm{d}v^2, \quad\text{(V.)}$$

so that $A\mathrm{d}u$ and $B\mathrm{d}v$ are elements of the arcs of these curves. The vector ρ being a function of u and v, we obtain additional relations connecting the six scalars $a', b', c', a_{,}, b_{,}, c_{,}$ by expressing that

$$\frac{\partial^2\rho}{\partial v\partial u}=\frac{\partial}{\partial v}(A\alpha)=\frac{\partial}{\partial u}(B\beta)=\frac{\partial^2\rho}{\partial u\partial v}. \quad\text{(VI.)}$$

Now, attending to (II.), we have for example, by (III.),

$$\mathrm{d}\alpha=\mathrm{V}(\omega'\mathrm{d}u+\omega_{,}\mathrm{d}v)\alpha=(\beta c'-\gamma b')\mathrm{d}u+(\beta c_{,}-\gamma b_{,})\mathrm{d}v, \quad\text{(VII.)}$$

and the differentials of β and γ are obtained by cyclically transposing $\alpha, \beta, \gamma, a', b', c', a_{,}, b_{,}, c_{,}$. Hence (VI.) at once leads to the three relations

$$\frac{\partial A}{\partial v}+Bc'=0,\quad \frac{\partial B}{\partial u}-Ac_{,}=0,\quad Ab_{,}+Ba'=0 \quad\text{(VIII.)}$$

obtained by equating the coefficients in

$$\frac{\partial A}{\partial v}\cdot\alpha+A(\beta c_{,}-\gamma b_{,})=\frac{\partial B}{\partial u}\beta+B(\gamma a'-\alpha c').$$

These three relations coupled with (IV.) give all that is necessary for the investigation.

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* Note that $\mathrm{V}\mathrm{d}qq^{-1}$ is not a perfect differential.

To ascertain the meaning of the scalars, observe that the vector curvatures of the curve u variable and v variable are (Art. 86, p. 134)

$$\frac{\partial\alpha}{\partial u}\cdot\frac{1}{A\alpha}=\frac{\beta b'+\gamma c'}{A},\quad \frac{\partial\beta}{\partial v}\cdot\frac{1}{B\beta}=\frac{\alpha a_,+\gamma c_,}{B},\quad\text{(IX.)}$$

so that by what we have shown $A^{-1}b'$ is the curvature of the normal section through u variable and $A^{-1}c'$ is the geodesic curvature of the same curve.

For any curve traced on the surface, if

$$\mathrm{U}d\rho=\mathrm{U}(\alpha A\mathrm{d}u+\beta B\mathrm{d}v)=\alpha\cos l+\beta\sin l,\ \cos l\mathrm{d}s=A\mathrm{d}u,\ \sin l\mathrm{d}s=B\mathrm{d}v,\quad\text{(X.)}$$

the vector curvature is

$$\frac{\mathrm{d}\,.\,\mathrm{U}\mathrm{d}\rho}{\mathrm{d}\rho}=\gamma\left(\frac{\partial l}{\partial s}+\frac{c'}{A}\cos l+\frac{c_,}{B}\sin l\right)$$
$$-\gamma\mathrm{U}\mathrm{d}\rho\left\{(a'\sin l-b'\cos l)\frac{\cos l}{A}+(a_,\sin l-b_,\cos l)\frac{\sin l}{B}\right\},\quad\text{(XI.)}$$

which follows easily on substituting for $\mathrm{d}u$ and $\mathrm{d}v$ in

$$\mathrm{d}\,.\,\mathrm{U}\mathrm{d}\rho=(\beta\cos l-\alpha\sin l)\mathrm{d}l+(\beta(c'\mathrm{d}u+c_,\mathrm{d}v)-\gamma(b'\mathrm{d}u+b_,\mathrm{d}v))\cos l$$
$$+(\gamma(a'\mathrm{d}u+a_,\mathrm{d}v)-\alpha(c'\mathrm{d}u+c_,\mathrm{d}v))\sin l.$$

Thus the geodesic curvature depends simply on c', $c_,$, and the rate of variation of the angle l which the curve makes with u variable. The normal curvature depends on the four quantities a', $a_,$, b', $b_,$. The relation (XI.) includes everything relating to the second differentials of the curve, and if we write for the curve $\alpha'=\mathrm{U}\mathrm{d}\rho$, $\gamma'=\mathrm{U}\,.\,\mathrm{d}\mathrm{U}\mathrm{d}\rho\,.\,\mathrm{d}\rho^{-1}$, $\gamma'\alpha'=\beta'$, we may, for brevity, replace (XI.) by the relation

$$\gamma'=\gamma\cos m+\gamma\alpha'\sin m,\quad\text{(XII.)}$$

and we may determine the torsion and everything depending on third differentials by differentiating once more.

Ex. 1. Determine the equations of the lines of curvature, and prove Gauss's theorem that the measure of curvature depends on differentials of the line element.

[If C' and $C_,$ are the principal curvatures, $\rho-C'^{-1}\gamma$ and $\rho-C_,^{-1}\gamma$ are the vectors to the centres of curvature, and expressing that these are stationary for the moment, we have

$$A\alpha\mathrm{d}u+B\beta\mathrm{d}v-C^{-1}(\alpha(b'\mathrm{d}u+b_,\mathrm{d}v)-\beta(a'\mathrm{d}u+a_,\mathrm{d}v))=0,$$

and according as we eliminate the ratio $\mathrm{d}u:\mathrm{d}v$ or C we have the equation of the lines of curvature, and the equation of the curvatures,

$$Aa'\mathrm{d}u^2+(Aa_,+Bb')\mathrm{d}u\mathrm{d}v+Bb_,\mathrm{d}v^2=0,\quad C^2AB-C(b'B-a_,A)+a'b_,-a_,b'=0.$$

By (IV.) and (VIII.) we see that the product of the curvatures is a function of A, B and their differential coefficients.]

Ex. 2. Prove that when the curves u and v are lines of curvature,

$$b'=C'A,\quad a_,=-C_,B,\quad a'=0,\quad b_,=0,\quad c'=-B^{-1}\frac{\partial A}{\partial v},\quad c_,=A^{-1}\frac{\partial B}{\partial u};$$

and show that

$$\frac{\partial C_,}{\partial u}=\frac{(C'-C_,)}{B}\frac{\partial B}{\partial u},\quad \frac{\partial C'}{\partial v}=\frac{(C_,-C')}{A}\frac{\partial A}{\partial v},$$

$$C'C_,=-\frac{1}{AB}\cdot\left\{\frac{\partial}{\partial v}\left(\frac{1}{B}\cdot\frac{\partial A}{\partial v}\right)+\frac{\partial}{\partial u}\left(\frac{1}{A}\cdot\frac{\partial A}{\partial u}\right)\right\}.$$

Ex. 3. If the curves u are geodesics, prove that we may take $A=1$, and that in this case

$$C'C_{\prime}=-\frac{1}{B}\cdot\frac{\partial^2 B}{\partial u^2},\quad G=\frac{\partial l}{\partial s}+\sin l\frac{1}{B}\cdot\frac{\partial B}{\partial u},$$

where G is the geodesic curvature of any curve, and l the angle it makes with the curve u variable.

[Here $c'=0$, so that A is independent of v, and by a change of the variable u we may put $A=1$.]

Ex. 4. Prove that the total curvature of any portion of the surface is

$$\iint C'C_{\prime}\mathrm{d}S=-\iint\frac{\partial^2 B}{\partial u^2}\mathrm{d}u\,\mathrm{d}v=-\int\frac{\partial B}{\partial u}\mathrm{d}v=\int\left(\frac{\partial l}{\partial s}-G\right)\mathrm{d}s,$$

where $\mathrm{d}S$ is an element of the surface; and where l is the angle the bounding curve makes with the curve u variable, G is the geodesic curvature of the bounding curve and $\mathrm{d}s$ an element of its length.

(a) Examine the case in which the bounding curve is composed of geodesics.

(iv) *Families of Curves and Surfaces.*

ART. **95.** If $\rho=\eta(t;\ a, b, c, \ldots)$, ...(I.)

where η is a given function of a variable parameter t and of certain scalar constants a, b, c, etc., the equation represents a family of curves, any particular member of the family being determined by assigning fixed values to the constants a, b, c, etc. If there are n constants, the family is said to be n-way, or to be of the n^{th} order.

The curves of the family which touch a given surface or intersect a given curve compose a family of order $n-1$.

If the given curve is $\rho=\eta_1(t_1)$, the condition of intersection

$$\eta(t;\ a, b, c, \ldots)=\eta_1(t_1) \quad \text{......................(II.)}$$

is equivalent to three scalar equations, so that on elimination of t and t_1 from these, we are left with a scalar equation in the constants a, b, c, etc., and thus one of the constants may be expressed in terms of the remaining $n-1$.

If the given surface is $f(\rho)=0$, the conditions for contact are

$$f(\eta)=0,\quad \frac{\partial f(\eta)}{\partial t}=0, \quad \text{......................(III.)}$$

and on elimination of t, a relation connecting the constants is obtained, so that a family of order $n-1$ touches the given surface.

ART. **96.** Expressing that an unknown surface $f(\rho)=0$ meets a curve of the family at the extremity of the vector ρ in n consecutive points we have

$$\rho=\eta,\quad S\nu\eta'=0,\quad S\nu\eta''+S\eta'\phi\eta'=0,$$
$$S\nu\eta'''+2S\eta''\phi\eta'+S\eta'\phi\eta''+S\eta'\phi_2(\eta'\eta')=0,\ \text{etc.,} \quad \text{........(I.)}$$

where the functions ϕ, ϕ_2, etc., are defined by the relations

$$d\nu=\phi d\rho,\ d^2\nu=\phi d^2\rho+\phi_2(d\rho, d\rho),\ \text{etc.} \qquad \text{(II.)}$$

The first of the equations (I.) is equivalent to three scalar equations, so that the system of equations is equivalent to $n+2$ scalar equations. We can from these eliminate t and the n constants a, b, c, etc., and the eliminant is a function of ρ, ν, ϕ, ϕ_2, etc., and is equivalent to the differential equation of surfaces met in n consecutive points by curves of the family.

In particular, the equation is equivalent to the differential equation of surfaces generated by curves of the family.

Ex. 1. Find the differential equation of surfaces generated by parallel lines.

[Here $\rho=\kappa+t\alpha$, $S\nu\alpha=0$, and the equation required is $S\nu\alpha=0$, α being a fixed vector and κ being arbitrary.]

Ex. 2. Find the differential equation of cones having a common vertex.

[In this case $\rho=\alpha+t\kappa$, $S\nu\kappa=0$, so that $S\nu(\rho-\alpha)=0$.]

Ex. 3. Prove that $SV\alpha\nu\phi V\alpha\nu=0$ is the differential equation of surfaces generated by lines perpendicular to the fixed vector α.

Ex. 4. The differential equation of surfaces generated by lines which meet the fixed line $V(\rho-\beta)\alpha=0$ is $SV\nu V(\rho-\beta)\alpha\,.\,\phi\,.\,V\nu V(\rho-\beta)\alpha=0$.

[If $\rho=\kappa+t\lambda$ is a generating line, $S(\kappa-\beta)\alpha\lambda=0$, $S\nu\lambda=0$, $S\lambda\phi\lambda=0$.]

Ex. 5. Find the differential equation of ruled surfaces.

[We have $S\nu\lambda=0$, $S\lambda\phi\lambda=0$, $S\lambda\phi_2(\lambda\lambda)=0$, and the equation is obtained by solving for λ (Art. 74, Ex. 3) from the first and second and substituting in the third.]

Ex. 6. Find the differential equation of surfaces generated by similar and similarly situated curves.

[Here a generating curve is $\rho=\kappa+a\alpha(t)$ where κ and a are constants to be eliminated and where $\alpha(t)$ is a given function of t.]

Ex. 7. The differential equation of surfaces generated by equal and similarly situated ellipses is

$$SVV\alpha\beta\,.\,\nu\,.\,\phi\,.\,VV\alpha\beta\,.\,\nu=(S\alpha\nu^2+S\beta\nu^2)^{\frac{3}{2}},$$

α and β being a pair of conjugate radii.

ART. **97.** As in the last article, being given the scalar equation of a family of surfaces involving n constants,

$$f(\rho;\ a, b, c, \ldots)=0, \qquad \text{(I.)}$$

we can determine the differential equation of a surface which at each point is touched by some member of the family in as many consecutive points as serve to eliminate the constants.

If only one constant is involved, only one surface is touched at each point by a member of the family, and that is the envelope obtained as the locus of intersection of consecutive members by eliminating the constant a between

$$f(\rho, a)=0 \ \text{ and } \ \frac{\partial f(\rho, a)}{\partial a}=0. \qquad \text{(II.)}$$

If two constants are involved, the conditions for contact with some unknown surface at the point ρ are

$$\nu = x\nu_0, \quad f(\rho\,;\ a, b) = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(III.)}$$

where ν is the normal to the unknown surface and ν_0 the normal to the surface of the family. The first equation, on elimination of the unknown scalar x, is equivalent to two scalar equations, and between these and the second we can eliminate a and b, and we obtain the differential equation of the touched surface as a function of ρ and ν, homogeneous in ν.

When the family contains three parameters, we express that the surfaces touch at two consecutive points, and we have

$$\nu = x\nu_0, \quad \phi\mathrm{d}\rho = x\phi_0\mathrm{d}\rho + \mathrm{d}x\,.\,\nu_0 = 0, \quad f(\rho;\ a, b, c) = 0, \quad \mathrm{S}\nu\mathrm{d}\rho = 0. \text{ (IV.)}$$

We can eliminate $\mathrm{d}\rho$ and replace the equations by

$$\nu = x\nu_0, \quad \mathrm{S}\nu(\phi - x\phi_0)^{-1}\nu = 0, \quad f(\rho\,;\ a, b, c) = 0\,; \ldots\ldots\ldots\text{(V.)}$$

and these equations are equivalent to five scalar equations from which to eliminate x, a, b and c.

Observe that we find two directions $\mathrm{d}\rho$ for contact according as we substitute one or other of the values of x given by the scalar equation (V.) in the second equation (IV.)

It is not hard to see that each additional condition of successive contact affords one additional scalar equation in x and the constants. In fact if we attend merely to the new unknowns $\mathrm{d}^m\rho$ and $\mathrm{d}^m x$ introduced in $\mathrm{d}^{m-1}(\phi\mathrm{d}\rho - x\phi_0\mathrm{d}\rho + \mathrm{d}x\nu_0) = 0$ and $\mathrm{d}^{m-1}\mathrm{S}\nu\mathrm{d}\rho = 0$, we see that they occur in the forms

$$\mathrm{d}^m\rho + (\phi - x\phi_0)^{-1}\nu_0\,.\,\mathrm{d}^m x + \text{etc.} = 0, \quad \mathrm{S}\nu\mathrm{d}^m\rho + \text{etc.} = 0\,;$$

and when we eliminate the vector $\mathrm{d}^m\rho$, the scalar $\mathrm{d}^m x$ disappears also by (V.). The preceding vector condition

$$\mathrm{d}^{m-2}(\phi\mathrm{d}\rho - x\phi_0\mathrm{d}\rho + \mathrm{d}x\nu_0) = 0$$

serves to eliminate $\mathrm{d}^{m-1}x$, and so on.

The conditions of contact at $n-1$ successive points serve to eliminate the n constants, and the result is the differential equation of surfaces touched at each point by some one member of the family in $n-1$ successive points. In particular, the equation is the differential equation of envelopes of the family obtained by replacing the n constants by *arbitrary functions* of a single constant.

When the family of surfaces is given in terms of two parameters t and u,

$$\rho = \eta(t, u\,;\ a, b, c, \ldots), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(VI.)}$$

we have $\nu = x\mathrm{V}\eta'\eta_,, \quad \mathrm{d}\nu = \phi(\eta'\mathrm{d}t + \eta_,\mathrm{d}u) = x\mathrm{d}\mathrm{V}\eta'\eta_, + \mathrm{V}\eta'\eta_,\mathrm{d}x, \ldots\text{(VII.)}$

and on direct elimination of $\mathrm{d}t$, $\mathrm{d}u$ and $\mathrm{d}x$,

$$\mathrm{S}\nu\,.\,[\phi\eta' - x\mathrm{V}(\eta''\eta_, + \eta'\eta_,')][\phi\eta_, - x\mathrm{V}(\eta_,'\eta_, + \eta'\eta_{,,})] = 0. \ldots\text{(VIII.)}$$

The next differentiation introduces d^2t, d^2u and d^2x, and these being eliminated by an equation analogous to (VIII.) we use (VII.) to get rid of dt, du and dx.

Ex. 1. Prove that for the envelopes of a family of spheres,

$$\pm U\nu R = \rho - \kappa,$$

where κ is the vector to the centre of a sphere and R the corresponding radius.

Ex. 2. The differential equation of envelopes of spheres of constant radius whose centres lie on a curve on the surface $f\rho = 0$ is $f(\rho \pm U\nu R) = 0$.

Ex. 3. The differential equation of the envelopes of spheres having their centres on the ellipse $\rho = \alpha \cos t + \beta \sin t$ is

$$(S\alpha\rho\nu)^2 + (S\beta\rho\nu)^2 = (S\alpha\beta\nu)^2.$$

Ex. 4. Find the differential equation of developable surfaces.

Ex. 5. Show how to find the differential equation of the envelopes of a surface carried parallel to itself.

[Take $\rho = \delta + \eta(t, u)$.]

Ex. 6. Find the envelopes of a rotated surface.

[Take $\rho = q \cdot \eta(t, u) \cdot q^{-1}$.]

ART. **98.** A differential equation of the first order presents itself in the form

$$F(\rho, \nu) = 0, \qquad \text{(I.)}$$

homogeneous in ν. For any variation of ρ and ν subject to this condition,

$$d \cdot F(\rho, \nu) = S\tau d\rho + S\mu d\nu = 0, \qquad \text{(II.)}$$

where τ and μ are determinate functions of ρ and ν. If the equation has a solution, there must be some scalar function of ρ, so that

$$d \cdot f\rho = nS\nu d\rho, \qquad \text{(III.)}$$

and for any arbitrary differentials of ρ, if $dn = S\sigma d\rho$,

$$d'df\rho = nSd'\nu d\rho + nS\nu d'd\rho + S\sigma d'\rho S\nu d\rho$$
$$= dd'f\rho = nSd\nu d'\rho + nS\nu dd'\rho + S\sigma d\rho S\nu d'\rho,$$

so that (compare Art. 91)

$$S(nd'\nu - \sigma S\nu d'\rho + \nu S\sigma d'\rho)d\rho - nSd'\rho d\nu = 0\,; \qquad \text{(IV.)}$$

and this general relation must include (II.) as a particular case.

Hence for some differential $d'\rho$ satisfying $S\nu d'\rho = 0$, we must have

$$x\tau = nd'\nu + \nu S\sigma d'\rho, \quad x\mu = -nd'\rho, \qquad \text{(V.)}$$

and from this we have the equivalent of Charpit's equations

$$\frac{d\rho}{\mu} = -\frac{V\nu d\nu}{V\nu\tau}, \quad S\nu d\rho = 0. \qquad \text{(VI.)}$$

EXAMPLES TO CHAPTER X.

Ex. 1. Determine the equations of the osculating circle and osculating helix of a curve in terms of the vectors α, β, γ and the scalars a_1 and c_1 corresponding to the point of contact, and find the deviation of the curve from the circle or helix.

Ex. 2. Show that the vector to a point on an ellipsoid may be expressed in the form

$$\rho = \alpha \cos u + \tau \sin u \text{ where } \mathrm{T}\tau = b,\ \mathrm{S}\lambda\tau = 0,\ \mathrm{T}\lambda = 1,$$

the vectors λ and α being constant but τ being variable.

(a) A tangential vector is

$$\mathrm{d}\rho = (-\alpha \sin u + \tau \cos u)\,\mathrm{d}u + \lambda\tau \sin u\mathrm{d}t,$$

and the equation of the tangent plane is

$$\mathrm{S}\nu\rho = b^2\mathrm{S}\lambda\alpha \text{ where } \nu = \mathrm{V}\lambda\tau\,(\alpha \sin u - \tau \cos u).$$

Ex. 3. The differential equation of a geodesic on the quadric $\mathrm{S}\rho\phi\rho + 1 = 0$ is

$$\mathrm{S}\phi\rho\mathrm{d}\rho\mathrm{d}^2\rho = 0.$$

(a) This equation, which expresses that $\phi\rho$, $\mathrm{d}\rho$ and $\mathrm{d}^2\rho$ are linearly connected, may by the aid of the differentials of the equation of the quadric be replaced by

$$\mathrm{d}^2\rho - \mathrm{d}\rho\frac{\mathrm{Sd}\rho\mathrm{d}^2\rho}{\mathrm{d}\rho^2} + \phi\rho\frac{\mathrm{Sd}\rho\phi\rho}{\phi\rho^2} = 0\,;$$

and operating by $\mathrm{S}\phi\mathrm{d}\rho$ an integrable relation,

$$\frac{\mathrm{Sd}^2\rho\phi\mathrm{d}\rho}{\mathrm{Sd}\rho\phi\mathrm{d}\rho} - \frac{\mathrm{Sd}\rho\mathrm{d}^2\rho}{\mathrm{d}\rho^2} + \frac{\mathrm{Sd}\rho\phi\rho}{\phi\rho^2} = 0,$$

is found which affords the integral

$$\mathrm{Sd}\rho\phi\mathrm{d}\rho\,.\,\phi\rho^2 = C\,.\,\mathrm{d}\rho^2.$$

(b) The geometrical interpretation is that PD is constant along the geodesic, where P is the central perpendicular on the tangent plane and where D is the diameter of the quadric parallel to the tangent to the geodesic. (Compare Ex. 14, p. 287.)

Ex. 4.* A unicursal curve of order n is represented by an equation of the form

$$\rho = \frac{(\alpha_0,\ \alpha_1,\ \alpha_2 \dots \alpha_n \between t,\ 1)^n}{(a_0,\ a_1,\ a_2 \dots a_n \between t,\ 1)^n}\,;$$

and in general this equation may be transformed into

$$\rho = \beta_0 + \Sigma_1{}^n \frac{\beta_m}{b_m - t},$$

and the curve may be described as the locus of the mean centre of corresponding points on n homographically divided lines.

(a) The equation of the asymptotic tangent parallel to β_1 is

$$\rho = \beta_0 + \Sigma_2{}^n \frac{\beta_m}{b_m - b_1} + u\beta_1.$$

Ex. 5. Find expressions for the curvature and torsion of a line of curvature on a quadric in terms of the elliptic coordinates of Art. 84.

Ex. 6. The vectors $\rho = \theta(t)$ to points on a curve are transformed by the operation of a linear vector function ϕ. Compare the curvature and torsion at corresponding points.

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* See *Proc. R.I.A.*, 3rd Series, Vol. iv., 1897.

Ex. 7. (*a*) If α, β, γ and δ are vectors from a common origin to four points A, B, C and D, it is always possible to determine four scalars a, b, c and d, so that
$$a\alpha+b\beta+c\gamma+d\delta=0.$$

(*b*) If the sum of these scalars is zero, the four points lie in a plane.

(*c*) It is also possible to determine a second set of scalars so that
$$a'\alpha^{-1}+b'\beta^{-1}+c'\gamma^{-1}+d'\delta^{-1}=0.$$

(*d*) If the sum of this new set vanishes, the points lie on a sphere passing through the origin.

(*e*) The equation of this sphere may be written in the form
$$S\rho^{-1}(\beta^{-1}\gamma^{-1}+\gamma^{-1}\alpha^{-1}+\alpha^{-1}\beta^{-1})=S\alpha^{-1}\beta^{-1}\gamma^{-1}.$$

(*f*) If it is possible to determine a third set of scalars so that
$$a''\alpha^{\frac{1}{2}}+b''\beta^{\frac{1}{2}}+c''\gamma^{\frac{1}{2}}+d''\delta^{\frac{1}{2}}=0,$$
the four vectors are edges of the right circular cone
$$SU\rho(U\,.\,\beta\gamma+U\,.\,\gamma\alpha+U\,.\,\alpha\beta)=SU\,.\,\alpha\beta\gamma.$$

(*g*) If the additional condition is imposed that the sum of the scalars of this third set vanishes, the four points lie on a surface whose equation may be written
$$S\lambda\rho^{\frac{1}{2}}=1,$$
λ being a constant vector.

(*h*) Discuss briefly the nature of this surface. (Bishop Law's Premium.)

Ex. 8. The differential equation of surfaces generated by lines of the complex (Art. 36, Ex. 4, p. 40)
$$f(\sigma,\tau)=0$$
may be found by eliminating σ and τ between this equation and
$$\sigma=V\rho\tau,\quad S\nu\tau=0,\quad S\tau\phi\tau=0.$$

(*a*) For the linear complex $S(\alpha\sigma+\beta\tau)=0$, the equation is
$$S\,.\,V\nu(V\alpha\rho+\beta)\phi V\nu(V\alpha\rho+\beta)=0.$$

(*b*) Lines common to the two linear complexes
$$S(\alpha\sigma+\beta\tau)=0,\quad S(\gamma\sigma+\delta\tau)=0,$$
generate the surfaces whose differential equation is
$$S\,.\,\nu(V\alpha\rho+\beta)(V\gamma\rho+\delta)=0.$$

(*c*) Find the differential equation of surfaces generated by lines of the congruency
$$f(\sigma,\tau)=0,\quad S(\alpha\sigma+\beta\tau)=0.$$

Ex. 9. If the vector β is a given function of a variable unit vector α, the equation
$$V(\rho-\beta)\alpha=0$$
represents a congruency of right lines.

(*a*) If $d\beta=\phi d\alpha$ determine the meaning of the several terms in the equation
$$\phi d\alpha+x d\alpha+\alpha dx=P\alpha d\alpha.$$

(*b*) A line of the congruency is intersected by consecutive lines at two *focal* points $\rho=\beta+x\alpha$ where x is a root of the quadratic
$$S\alpha(\phi+x)^{-1}\alpha=0,\quad\text{or}\quad S\alpha(x^2+x\chi_0+\psi_0)\alpha-S\epsilon\alpha^2=0,$$
ϵ being the spin-vector of ϕ and ϕ_0 being the self-conjugate part.

(*c*) The points of closest approach of consecutive rays to the ray $\rho=\beta+x\alpha$ lie between the *extreme* points determined by the condition that $SUd\alpha\phi Ud\alpha$ may be a maximum, and the corresponding values of x are the roots of the quadratic

$$S\alpha(\phi_0+x)^{-1}\alpha=0, \quad \text{or} \quad S\alpha(x^2+x\chi_0+\psi_0)\alpha=0.$$

(*d*) The vectors of shortest distance at the extreme points between the ray α and its consecutives are mutually perpendicular; and if these shortest vectors are parallel to the unit vectors α' and $\alpha_{,}$, the extreme points are determined by $$x'=S\alpha_{,}\phi\alpha_{,} \quad \text{and} \quad x_{,}=S\alpha'\phi\alpha'.$$

(*e*) If the vectors α, α' and $\alpha_{,}$ are in positive order of rotation so that $\alpha'\alpha_{,}=\alpha$, $$S\alpha'\phi\alpha_{,}=-S\alpha_{,}\phi\alpha'=-S\epsilon\alpha\,;$$

and if the shortest vector at the point corresponding to x makes the angle u with α' so that $$U\alpha d\alpha=\alpha'\cos u+\alpha_{,}\sin u,$$

the scalars x and P are connected with x' and $x_{,}$ by the relations,

$$x=x'\cos^2 u+x_{,}\sin^2 u, \quad P=S\epsilon\alpha+(x_{,}-x')\sin u\cos u.$$

Ex. 10. A circle may be represented by means of a pair of vectors (κ, λ) since its equations may be thrown into the form

$$T(\rho-\kappa)=T\lambda, \quad S\lambda(\rho-\kappa)=0\,;$$

and an equation such as $$f(\kappa, \lambda)=0,$$

where f is a general function, may be regarded as representing a family of circles.

(*a*) In like manner an equation such as

$$f(\alpha, \beta, \gamma)=0 \quad \text{where} \quad S\alpha\beta=0$$

represents a family of conics, γ being the vector to the centre of one of the conics and α and β being its principal vector radii. (Compare Ex. 11, p. 103.)

Ex. 11. The general surface generated by a variable circle (κ, λ) may be represented by

$$\rho=\kappa+\lambda\tau \quad \text{where} \quad S\lambda\tau=0, \quad T\tau=1,$$

the vectors κ and λ being functions of a single parameter and the auxiliary vector τ being arbitrary so far as the conditions allow.

(*a*) If P is a scalar analogous to the parameter of distribution of a ruled surface, $$P\frac{d\tau}{\tau}=d\kappa+d\,.\,\lambda\tau. \quad \text{Hence } d\tau=\frac{\tau d\lambda-d\kappa}{P\tau-\lambda},$$

and because $Sd\tau=0$, $S\tau d\tau=0$, we find

$$P=\frac{S(d\kappa-\tau d\lambda)\lambda}{S\tau d\kappa}=\frac{S(d\lambda+\tau d\kappa)\lambda}{S\tau d\lambda}.$$

(*b*) These expressions for P lead to four values of the vector τ which determine points at which neighbouring elements of successive circles approach most closely or are most widely separated.

(*c*) If successive circles intersect in one point

$$T(Vd\lambda\lambda Sd\lambda\lambda+Vd\kappa\lambda Sd\kappa\lambda)=T\,.\,\lambda S\lambda d\lambda d\kappa$$

and the vector to the point of intersection is

$$\rho=\kappa+\frac{Vd\lambda\lambda Sd\lambda\lambda+Vd\kappa\lambda Sd\kappa\lambda}{S\lambda d\lambda d\kappa}.$$

(*d*) If successive circles intersect in two points, the vector just found becomes indeterminate, and

$$\mathrm{V}d\lambda\lambda\mathrm{S}d\lambda\lambda+\mathrm{V}d\kappa\lambda\mathrm{S}d\kappa\lambda=0\,;$$

and when this condition is satisfied, the surface may be generated by the motion of the sphere,

$$\mathrm{T}\left(\rho-\kappa+\lambda\frac{\mathrm{S}\lambda d\lambda}{\mathrm{S}\lambda d\kappa}\right)=\mathrm{T}\lambda\left(1+\frac{\mathrm{S}\lambda d\lambda^2}{\mathrm{S}\lambda d\kappa^2}\right)^{\frac{1}{2}}.$$

(*e*) In the general case, the equation of a normal to the surface is

$$\mathrm{V}\,.\,(\rho-\kappa-\lambda\tau)\mathrm{V}\tau(d\kappa+\mathrm{V}d\lambda\,.\,\tau)=0\,;$$

and when this is expanded we obtain two scalar equations which combined with the equations of condition enable us to eliminate τ, so that we find the equation of the surface generated by the normals along the circle (κ, λ) to be

$$\mathrm{S}(\rho-\kappa)d\kappa-\mathrm{S}\lambda d\lambda\pm\mathrm{S}(\rho d\lambda-\kappa d\lambda-d\kappa\lambda)\mathrm{UV}(\rho-\kappa)\lambda=0.$$

This surface is of the fourth order, and normals at the extremities of diameters of the circle intersect in a nodal conic.

CHAPTER XI.

STATICS.

ART. **99**. If α is the vector to the point of application of a force which is represented in magnitude and direction by the vector β, the moment of the force with respect to the origin is $V\alpha\beta$—the vector area of the parallelogram determined by α and β; and the moment about the extremity of the vector γ is $V(\alpha-\gamma)\beta$. The force may be replaced by an equal force β at the origin, and a couple $V\alpha\beta$; or by an equal force β at the extremity of the vector γ and a couple $V(\alpha-\gamma)\beta$.

For any number of forces, the quaternion quotient of the resultant vector moment at the origin by the resultant force is (*Elements*, Art. 416 (11))

$$q=\frac{\Sigma V\alpha\beta}{\Sigma\beta}=p+\varpi \quad \text{where} \quad p=Sq, \quad \varpi=Vq; \quad \text{.........(I.)}$$

and because $\quad \Sigma V\alpha\beta=p\Sigma\beta+\varpi\Sigma\beta=p\Sigma\beta+V\rho\Sigma\beta, \quad$ ....(II.)

if ρ is the vector to any point on the line represented by

$$V\rho\Sigma\beta=\varpi\Sigma\beta=(\Sigma\beta)^{-1}V\,.\,\Sigma\beta\Sigma V\alpha\beta, \quad \text{..............(III.)}$$

we may replace the system of forces by a force $\Sigma\beta$ acting along the line (III.) and by a couple $p\Sigma\beta$ having its axis parallel to that line. This is the reduction to Poinsot's *central axis.*

The system of forces constitute a *wrench* upon a screw;* the scalar p, which is independent of the origin, is the *pitch* of the screw, and the vector ϖ is the perpendicular from the origin on the axis of the screw—Poinsot's central axis.

If the resultant reduces to a single force, p is zero or $S\Sigma V\alpha\beta(\Sigma\beta)^{-1}=0$; and if they reduce to a couple $\Sigma\beta=0$ and p is infinite. If the forces equilibrate

$$\Sigma\beta=0, \quad \Sigma V\alpha\beta=0. \quad \text{.........................(IV.)}$$

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* Sir Robert S. Ball, *Treatise on the Theory of Screws*, Cambridge, 1900.

Hamilton uses a second quaternion

$$Q=\frac{\Sigma\alpha\beta}{\Sigma\beta}=\frac{S\Sigma\alpha\beta}{\Sigma\beta}+q=p+\gamma, \qquad \text{(v.)}$$

and the scalar of this quaternion is the pitch while the vector terminates at a point which is independent of the origin—the Hamiltonian *centre of the system of forces.* This point is evidently situated on the central axis (III.).

The quaternion $\alpha\beta$ is called by Hamilton the *quaternion moment* of the force β with respect to the origin. Its vector part is the moment of the force and its scalar part is minus the *virial.* We shall write for any number of forces

$$\Sigma\alpha\beta=\mu+m'', \quad \Sigma\beta=\lambda, \qquad \text{(vi.)}$$

so that we have

$$\mu=V\Sigma\alpha\beta=p\lambda+\varpi\lambda, \quad \gamma=\varpi+m''\lambda^{-1}, \qquad \text{(vii.)}$$

where μ is the resultant vector moment at the origin and where m'' is minus the resultant virial at the same point. The *plane of no virial* is represented by

$$S\Sigma(\alpha-\rho)\beta=0 \quad \text{or} \quad S\rho\lambda=m''; \qquad \text{(viii.)}$$

and Hamilton's centre is obviously the intersection of this plane and the central axis.

Ex. 1. Vectors (α) are drawn from a *variable* origin to the points of application of forces (β). The equation

$$\Sigma V\alpha\beta=0$$

implies equilibrium.

[If the vectors α_0 are drawn from a *fixed* origin to the points of application, we must have separately $\Sigma\beta=0$, $\Sigma V\alpha_0\beta=0$ (*Elements*, Art. 416).]

Ex. 2. Forces act at the vertices of a triangle, in its plane and proportional and perpendicular to the opposite sides. Prove that they are in equilibrium.

[If α, β and γ are the vectors from a *variable* origin, the forces are $\nu(\beta-\gamma)$, $\nu(\gamma-\alpha)$, $\nu(\alpha-\beta)$ where ν is a vector perpendicular to the plane of the triangle. The moment formed as in the last example vanishes identically because $V\alpha\nu\beta=V\beta\nu\alpha$, etc.]

Ex. 3. The conditions of equilibrium of a rigid body may be expressed by the equation

$$\Sigma S\beta d\alpha=0,$$

which contains the principle of virtual velocities (*Elements*, Art. 416 (17)).

[For any possible small displacement of the body $d\alpha=\delta+V\omega\alpha$ where δ and ω are arbitrary. Hence $\Sigma\beta=0$, $\Sigma V\alpha\beta=0$.]

Ex. 4. The moment of the force AB about the line CD is six times the volume of the tetrahedron ABCD divided by the number of units of length in CD.

[The vector moment at the point C is $V.CA.AB$ and the component along CD is $-S(UCD.V.CA.AB)=-S.CD.CA.ABT.CD^{-1}$.]

Ex. 5. A force of unit intensity acts along the line $V(\rho-\alpha)\beta=0$. Its moment about the line $V(\rho-\alpha')\beta'=0$ is $-S(\alpha-\alpha')U\beta\beta'$.

Ex. 6. If three forces are in equilibrium, they must be in the same plane.

[Operate on the condition $V(\rho-\alpha)\beta+V(\rho-\alpha')\beta'+V(\rho-\alpha'')\beta''=0$ by $S(\rho-\alpha)$ and put $\rho=\alpha'$ where we find $S(\alpha'-\alpha)(\alpha'-\alpha'')\beta''=0$.

Ex. 7. If four forces are in equilibrium, their lines of action are generators of a hyperboloid.

[One method of proof (Chap. VIII., Ex. 10, p. 103) is to express the four vector moments $V\alpha_n\beta_n$, etc., in terms of the four forces by means of a linear vector function, so that $V\alpha_n\beta_n=\phi\beta_n+\omega$. The vector ω is zero because $\Sigma V\alpha_n\beta_n=0$, $\Sigma\beta_n=0$, and therefore the equation of a line of action is $\rho=\phi\beta_n\beta_n^{-1}+x\beta_n$. (See Art. 79, p. 116.)]

Ex. 8. Resolve a wrench into forces along the edges of a tetrahedron ABCD.

[If μ is the moment and λ the force of the given wrench at the fixed origin of vectors O, the moment at the point P is

$$\mu-V\,.\,\mathrm{OP}\,.\,\lambda=\Sigma t_{\mathrm{AB}}V\,.\,\mathrm{PA}\,.\,\mathrm{AB}$$

where t_{AB}, etc., are scalars proportional to the forces along the edges. Take the point P at D, and

$$\mu-V\,.\,\mathrm{OD}\,.\,\lambda=t_{\mathrm{AB}}\,.\,V\,.\,\mathrm{DA}\,.\,\mathrm{AB}+t_{\mathrm{BC}}V\,.\,\mathrm{DB}\,.\,\mathrm{BC}+t_{\mathrm{CA}}V\,.\,\mathrm{DC}\,.\,\mathrm{CA}$$

serves to determine three of the unknown scalars. Operate by $S\,.\,\mathrm{DC}$ and

$t_{\mathrm{AB}}\,.\,S\,.\,\mathrm{DA}\,.\,\mathrm{DB}\,.\,\mathrm{DC}=S(\mu-V\,.\,\mathrm{OD}\,.\,\lambda)\mathrm{DC}$, or $t_{\mathrm{AB}}\,.\,(\mathrm{ABCD})=S\,.\,\mathrm{CD}\,.\,\mu+S\,.\,\mathrm{OC}\,.\,\mathrm{OD}\,.\,\lambda$.]

ART. **100.** To reduce a system of forces to two forces, let μ and λ be the resultant couple at the origin and the resultant force of the system, and assume

$$\mu=V\alpha\beta+V\alpha'\beta',\quad \lambda=\beta+\beta',\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(I.)}$$

where β and β' are the two forces and α and α' the vectors to their points of application. Hence

$$\beta'=\lambda-\beta,\quad \mu=V(\alpha-\alpha')\beta+V\alpha'\lambda\,;\ \ldots\ldots\ldots\ldots\text{(II.)}$$

and from the form of the second equation, it is obvious that if two of the unknown vectors α, α', β are suitably assumed, the third may be regarded as the vector to a point on a determinate line. But a condition must be satisfied, for on operating in turn by $S(\lambda-\beta)$ and $S(\alpha-\alpha')$ we have

$$S(\lambda-\beta)\mu=S\lambda\alpha\beta\quad\text{and}\quad S(\alpha-\alpha')\mu=S\alpha\alpha'\lambda,\ \ldots\ldots\text{(III.)}$$

so that if any one of the three unknown vectors is assumed (say α) the other two may be regarded as terminating on definite planes. Suitably selecting either β or α' in accordance with (III.) (which is a consequence of (II.)), the remaining vector is constrained by (II.) to terminate on a line.

Ex. 1. A rigid body is acted on by any number of forces. It is required to equilibrate the body by two forces whose points of application are situated on given lines.

[If ξ and ξ' are the required forces and $V(\rho-\alpha)\beta=0$, $V(\rho-\alpha')\beta'=0$ the equations of the given lines, we have

$$\lambda+\xi+\xi'=0,\ \ \mu+V(\alpha+x\beta)\xi+V(\alpha'+x'\beta')\xi'=0,$$

where x and x' are scalars. Hence

$$S(\alpha-\alpha'+x\beta-x'\beta')\mu-S(\alpha+x\beta)(\alpha'+x'\beta')\lambda=0,$$

and this equation of condition establishes a homography connecting the points of application.]

Ex. 2. A framework is composed of rods jointed by smooth hinges. Three of the rods, A_4A_1, A_4A_2 and A_4A_3 terminate at a point A_4 and are acted on by given wrenches. Determine the reactions at the joints; it being supposed that the three rods are not coplanar.

[Let (μ_{mn}, λ_{mn}) represent the wrench applied to the rod A_mA_n, the origin of vectors being taken as base-point, and let β_{mn} be the reaction of the joint on the rod at the point A_m. For equilibrium of the rod A_4A_1,

$$\mu_{41}-V\rho\lambda_{41}+V(\alpha_4-\rho)\beta_{41}+V(\alpha_1-\rho)\beta_{14}=0,$$

and putting $\rho=\alpha_1$, this gives

$$\mu_{41}-V\alpha_1\lambda_{41}+V(\alpha_4-\alpha_1)\beta_{41}=0,$$

or, for some scalar x_{41},

$$\beta_{41}=(\mu_{41}-V\alpha_1\lambda_{41}+x_{41})(\alpha_4-\alpha_1)^{-1}.$$

For equilibrium of the joint A_4, we have $\beta_{41}+\beta_{42}+\beta_{43}=0$, or

$$\Sigma_1^3(\mu_{4n}-V\alpha_n\lambda_{4n})(\alpha_4-\alpha_n)^{-1}=-\Sigma_1^3x_{4n}(\alpha_4-\alpha_n)^{-1}$$

and from this vector equation the three scalars x_{4n} can be found.]

Ex. 3. A rigid body is in equilibrium under the action of an impressed system of forces (μ, λ) and the tensions of two strings $A'A$ and $B'B$ attached to points A' and B' in the body and to fixed points A and B. Show that the forces exerted by the strings on the body are represented by

$$x(\alpha-\alpha')=\frac{\mu+\lambda\beta+t}{\alpha-\beta},\ \ y(\beta-\beta')=\frac{\mu+\lambda\alpha+t}{\beta-\alpha}$$

where x, y and t are scalars which may be determined by expressing that the lengths of the lines $A'A$, $B'B$, $A'B'$ and AB are given, and where α, β, α' and β' are the vectors from the base-points to the points A, B, A' and B'.

(*a*) What condition is implied in these equations?

(*b*) If a, b, c and d are the tensors of the vectors $A'A$, $B'B$, $A'B'$ and AB, respectively, show that the scalar t satisfies the equation

$$c=T\{aU(\mu+\lambda\beta+t)+bU(\mu+\lambda\alpha+t)+d\}.$$

ART. **101**. The resultant quaternion moment (Art. 99 (VI.)) for an arbitrary base-point (the origin of the vectors α) of a system of forces (β) acting at points fixed in a rigid body is the first quaternion invariant* of the linear vector function

$$\phi\rho=\Sigma\alpha S\beta\rho, \quad\text{..............................(I.)}$$

the first scalar invariant of this function being minus the resultant virial $(m''=\Sigma S\alpha\beta)$, and double the spin-vector being the resultant vector moment $(\mu=\Sigma V\alpha\beta)$.

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*That is the invariant $-\phi i\,.\,i-\phi j\,.\,j-\phi k\,.\,k$. Compare Art. 67, Ex. 7, p. 97.

If the forces receive a common conical rotation round their points of application so that each vector β is replaced by $q\beta q^{-1}$, the function $\phi\rho$ changes into $\phi(q^{-1}\rho q)$; and if the body is rotated so that α becomes $q\alpha q^{-1}$, the function becomes $q(\phi\rho)q^{-1}$. The results of Art. 70 show that there are four rotations applicable either to the body or to the forces which render the function self-conjugate;* and in this case the resultant is a single force passing through the origin. These four positions of the body relative to the forces are called the *initial positions.*

If $\lambda(=\Sigma\beta)=0$, the resultant is a couple for all relative positions. If the forces are in *astatic equilibrium*, the couple (as well as the resultant force) must vanish for all rotations; but this can only happen when the function ϕ vanishes identically because a function such as $q(\phi\rho)q^{-1}$ cannot be self-conjugate for all quaternions q. Thus the necessary and sufficient conditions for astatic equilibrium are

$$\phi=0, \quad \lambda=0; \quad \text{(II.)}$$

and these are equivalent to twelve scalar relations connecting the forces and the points of application.

In general reduction of the function ϕ to a trinomial form

$$\phi\rho=\gamma_1 S\lambda_1\rho+\gamma_2 S\lambda_2\rho+\gamma_3 S\lambda_3\rho, \quad \lambda_1+\lambda_2+\lambda_3=\lambda, \quad \text{(III.)}$$

in which λ_1 and λ_2 are arbitrarily assumed, corresponds to the reduction of the system of forces to three forces λ_1, λ_2 and λ_3 *astatically equivalent* to the given system; and it is easy to see that the points of application of these forces, the extremities of the vectors $\gamma_1=\phi V\lambda_2\lambda_3 : S\lambda_1\lambda_2\lambda_3$, etc., are fixed relatively to the body and lie in the *central plane*

$$S\rho\psi'\lambda=m \quad \text{or} \quad S\rho\phi'^{-1}\lambda=1. \quad \text{(IV.)}$$

Reduction of the function to the standard form of Art. 70 gives a particularly simple set of equivalent forces or couples.

The vector $\phi\lambda$ is obviously fixed in the body, and when the origin is transferred to the extremity of the vector $\phi\,.\,\lambda^{-1}$ the linear function (which we continue to denote by ϕ) corresponding to this special origin—the *astatic centre*—satisfies the condition

$$\phi\lambda=0. \quad \text{(V.)}$$

As one root of ϕ is now zero, the function is reducible to the binomial form, and the auxiliary ψ function is of the type

$$\psi\rho=\lambda S\kappa\rho \quad \text{(VI.)}$$

where κ is a vector fixed in the body. The equation of the central plane is now $S\kappa\rho=0$.

* These are the rotations which convert i', j', k' of the article cited into $+i, +j, +k$; $+i, -j, -k$; $-i, +j, -k$; or $-i, -j, +k$. Compare the footnote to the article cited.

In addition to the equations (v.) and (vi.) we have

$$\phi'\lambda = V\lambda\mu \quad \text{and} \quad \phi\mu = \phi'\mu = V\kappa\lambda\,; \qquad \text{(vii.)}$$

the first is obvious because μ is double the spin-vector and the second follows from Art. 68 because $-\phi\mu$ is double the spin-vector of ψ. These relations coupled with the expression

$$\mu = p\lambda + V\eta\lambda, \qquad \text{(viii.)}$$

for the moment in terms of the pitch p and the vector η from the astatic centre to a point on the central axis of the forces in any position enable us to deduce all the theorems of astatics. We first remark that *the function* $\phi\phi'$ *is fixed relatively to the body* (or to the vectors α) and that *the function* $\phi'\phi$ *is fixed relatively to the vectors* β (or to the directions of the forces).

In order to determine the arrangement of the central axes relatively to the forces, operate on (viii.) by the function ϕ, and by (vii.) we find

$$\phi V\eta\lambda = V\kappa\lambda, \qquad \text{(ix.)}$$

so that $\quad T\phi V\eta\lambda = TV\kappa\lambda \quad \text{or} \quad SV\eta\lambda\phi'\phi V\eta\lambda = (V\kappa\lambda)^2\,; \qquad \text{(x.)}$

and therefore *relatively to the forces the central axes compose a coaxial family of similar elliptic cylinders* whose linear dimensions are proportional to the cosine of the inclination ($TVU\kappa\lambda$) of the central plane to the axes whose direction ($U\lambda$) is of course fixed relatively to the forces.

The arrangement of the central axes in the body is determined by the equation

$$\phi'\lambda = V\lambda V\eta\lambda \qquad \text{(xi.)}$$

obtained by operating on (viii.) by $V\lambda$ and attending to (vii.). Taking the tensor

$$TV\eta\lambda = T\phi' U\lambda = \sqrt{(-SU\lambda\phi\phi' U\lambda)}\,; \qquad \text{(xii.)}$$

and the *locus of central axes having a given direction* $U\lambda$ *relatively to the body is a right circular cylinder* whose radius is the reciprocal of the parallel radius of the elliptic cylinder

$$T\phi'\rho = T\lambda \quad \text{or} \quad S\rho\phi\phi'\rho = \lambda^2. \qquad \text{(xiii.)}$$

To each generator of a cylinder (x.) corresponds one of the cylinders (xii.) which is traced out by that generator when the forces are rotated round the vector λ. In terms of the vectors σ and τ of Art. 36, Ex. 4 ($\tau \parallel \lambda$), we may replace (xii.) by

$$T\lambda T\sigma = T\phi'\tau, \qquad \text{(xiv.)}$$

and this equation represents a complex of the second order—*the assemblage of lines in the body which become central axes by suitable rotation of the forces.*

We shall now determine the pitch corresponding to each central axis. Operating by ϕ' on (viii.) we have by (vii.)

$$p\phi'\lambda + \phi' V\eta\lambda = V\kappa\lambda, \qquad \text{(xv.)}$$

and operating on this by $S\phi'\lambda$ or $SV\lambda\mu$ or $SV\lambda V\eta\lambda$ we deduce

$$pT\phi'\lambda^2 - S\lambda\phi\phi' V\eta\lambda = T\lambda^2 S\kappa\eta\lambda. \qquad \text{(xvi.)}$$

This equation gives p in terms of the vectors determining the central axes. Again we obtain an equivalent expression by taking the tensor of (xv.), and on replacing λ by τ and $V\eta\lambda$ by σ the result is

$$p^2 T\phi'\tau^2 - 2pS\tau\phi\phi'\sigma + T\phi'\sigma^2 = TV\kappa\tau^2. \qquad \text{(xvii.)}$$

This represents a complex of the second order and the lines common to the complex (xiv.) compose a congruency of the fourth order and the fourth class—*the assemblage of lines in the body which become axes of screws of given pitch for suitable rotation of the forces.**

Since (XVI.) is linear in η, it represents a plane when the direction of λ is given which cuts the cylinder (XII.) in two axes corresponding to the given pitch. The plane touches the cylinder if

$$p\mathrm{T}\phi'\lambda\mathrm{T}\lambda^2 = \pm\mathrm{TV}\lambda(\phi\phi'\lambda + \kappa\mathrm{T}\lambda^2), \qquad \text{(XVIII.)}$$

and this relation determines the limiting values of the pitch for a given direction $\mathrm{U}\lambda$.

The function $\phi_\eta\rho$ corresponding to an arbitrary base-point—the extremity of the vector η—is

$$\phi_\eta\rho = \phi\rho - \eta\mathrm{S}\lambda\rho \qquad \text{(XIX.)}$$

because $\phi\rho$ is of the form $\Sigma\alpha\mathrm{S}\beta\rho$. The function $\phi_\eta\phi_\eta'$ for this base-point is

$$\phi_\eta\phi_\eta'\rho = \phi\phi'\rho - \mathrm{T}\lambda^2 . \eta\mathrm{S}\eta\rho\,; \qquad \text{(XX.)}$$

and supposing u^2 to be a latent root and α to be a unit vector along the corresponding axis, it appears on inversion of the function $\phi\phi' - u^2$ that the latent roots (u^2, u'^2, u''^2) of $\phi_\eta\phi_\eta'$ are parameters of the quadrics of the confocal system (fixed in the body)

$$\mathrm{S}\rho(\phi\phi' - u^2)^{-1}\rho\mathrm{T}\lambda^2 = 1 \qquad \text{(XXI.)}$$

which pass through the extremity of η, and that the axes $(\alpha, \alpha', \alpha'')$ of the function are the normals to these confocals. Reduction of ϕ_η to the standard form of Art. 70 gives

$$\phi_\eta\rho = u\alpha\mathrm{S}\beta\rho + u'\alpha'\mathrm{S}\beta'\rho + u''\alpha''\mathrm{S}\beta''\rho \qquad \text{(XXII.)}$$

where the unit vectors β are likewise mutually perpendicular so that the system of forces may be replaced by λ acting at the extremity of η and by three couples (such as that due to the unit force $+\beta$ acting at the extremity of $\eta + \frac{1}{2}u\alpha$ and $-\beta$ acting at the extremity of $\eta - \frac{1}{2}u\alpha$) whose arms $(u\alpha, u'\alpha', u''\alpha'')$ are mutually perpendicular as well as the forces (β, β', β'').

The parameters of the confocals (XXI.) touched by an arbitrary line (σ, τ) are the roots of the quadratic equation (Art. 83, Ex. 2, p. 124).

$$\mathrm{S}\tau\{\psi'\psi - u^2(M'' - \phi\phi') + u^4\}\tau + \mathrm{S}\sigma(\phi\phi' - u^2)\sigma . \mathrm{T}\lambda^2 = 0$$

where M'' is the first invariant of $\phi\phi'$, observing that in general the ψ function of $\phi\phi'$ is $\psi'\psi$; or of the equation

$$u^4\mathrm{T}\tau^2 - u^2(M''\mathrm{T}\tau^2 - \mathrm{T}\phi'\tau^2 + \mathrm{T}\sigma^2\mathrm{T}\lambda^2) + \mathrm{T}\psi\tau^2 + \mathrm{T}\phi'\sigma^2\mathrm{T}\lambda^2 = 0\,; \qquad \text{(XXIII.)}$$

and when the line belongs to the complex of central axes (XIV.) the equation reduces by (XVII.) without much trouble to

$$u^4 - M''u^2 + M' = p^2\mathrm{T}\phi'\mathrm{U}\tau^2 + 2p\mathrm{S}\tau^{-1}\phi\phi'\sigma \qquad \text{(XXIV.)}$$

where $M'(=\mathrm{T}\kappa^2\mathrm{T}\lambda^2)$ is the second invariant of $\phi\phi'$ or the first of $\psi'\psi$. This shows that *the central axes touch confocals having the sum of their parameters constant* and equal to M''; and in particular we have Minding's theorem for $p=0$ that *the lines of action of single force resultants intersect the focal conics* of the system (XXI.) since the parameters of the touched confocals are in this case the finite latent roots of $\phi\phi'$ and the focal conics obviously correspond to these parameters. The theorem respecting the constant sum of para-

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* The former equation (XVI.) in terms of τ and σ is

$$p\mathrm{T}\phi'\tau^2 - \mathrm{S}\tau\phi\phi'\sigma = \mathrm{T}\lambda\mathrm{T}\tau\mathrm{S}\kappa\sigma\,;$$

and on rationalization this is seen to represent a complex of the fourth order, and it may be shown that coupled with (XIV.) it reduces to (XVII.) affected by the factor $\mathrm{T}\phi'\tau^2$.

meters is otherwise deducible from Art. 83, Ex. 3, for the cone of lines of the complex (XIV.) through the extremity of the vector η is expressible in the form

$$S\tau(\phi\phi' - T\eta^2 T\lambda^2 - \eta T\lambda^2 . S\eta)\tau = 0. \qquad \text{(xxv.)}$$

Moreover (Art. 83 (x.)) this is the reciprocal of the tangent cone to the confocal (XXI.) whose parameter is $u^2 = T\eta^2 T\lambda^2$. According as the tangent cone becomes more and more obtuse by variation of the vector η and finally becomes a tangent plane, the reciprocal cone becomes more and more acute and finally coincides with the normal to the quadric, and the locus of such points is the surface

$$S\eta(\phi\phi' - T\eta^2 T\lambda^2)^{-1}\eta T\lambda^2 = 1. \qquad \text{(xxvi.)}$$

This surface is a quartic analogous to Fresnel's wave-surface, and its equation may be reduced to the form

$$T\eta = \frac{TV\kappa\eta}{T\phi'\eta} = \frac{TV\kappa\eta}{T\phi' . \kappa^{-1}V\kappa\eta} = \frac{T\kappa}{T\phi' U . \kappa^{-1}V\kappa\eta}, \qquad \text{(xxvii.)}$$

remembering that $\phi'\kappa = 0$. In this form it is apparent that the surface consists of a system of circles concentric with the astatic centre, coplanar with the vector κ and of radius proportional to that of the elliptic cylinder (XIII.) which is parallel to the radius in the central plane. For points inside this surface the cones of axes are imaginary.

The boundary of the region containing the feet of central perpendiculars on the axes has been investigated by Tait (*Quaternions*, Art. 403).

Expressing that $T\eta$ is a maximum when $U\eta$ is given and when τ is subject to the conditions (xxv.)

$$S\eta\tau = 0, \quad S\tau(\phi\phi' - T\eta^2 T\lambda^2)\tau = 0,$$

the equation of the boundary is found to be

$$S\eta(\phi\phi' - T\eta^2 T\lambda^2)^{-1}\eta = 0\,; \qquad \text{(xxviii.)}$$

and this represents a surface of the sixth order analogous to the inverse of a Fresnel's wave-surface, and on expansion it affords a quadratic in $T\eta^2$ corresponding to any given value of $U\eta$ whose roots are the limiting values of the squares of the perpendiculars.

Ex. If vectors are drawn in the body from an arbitrary base-point to represent the resultant moment, the locus of their extremities is an ellipse when the forces receive all possible rotations about a given axis.*

[Here $\mu = V\Sigma\alpha q\beta q^{-1} = V\Sigma\alpha(1+t\iota)\beta(1+t\iota)^{-1}$ where t is the tangent of half the angle of rotation and where ι is a unit vector along the axis of rotation, and the form of this equation establishes the theorem.]

ART. **102**. The resultant of any system of forces has been reduced in Art. 99 to a *wrench* which may be denoted by the symbol (μ, λ) where

$$\mu = p\lambda + V\eta\lambda \qquad \text{(I.)}$$

is the resultant moment with respect to the origin, where p is the pitch, where η is the vector to any point on the axis and where λ is the resultant force. The wrench $(t\mu, t\lambda)$, where t is any scalar, has by (I.) the same pitch and the same axis as (μ, λ). It is therefore said to be a wrench on the same *screw* as (μ, λ) and it may be denoted by $t(\mu, \lambda)$. The *intensity* of a wrench is

* See Joly, *Trans. R.I.A.*, Vol. xxxii., pp. 218 *et seq.*

the magnitude of a resultant force ($T\lambda$), and the wrench ($t\mu$, $t\lambda$) has t-fold the intensity of (μ, λ).

It is often necessary to compound wrenches situated upon different screws, and we shall investigate the simplest expression for the wrench

$$(\mu, \lambda) = t_1(\mu_1, \lambda_1) + t_2(\mu_2, \lambda_2) + t_3(\mu_3, \lambda_3) \quad \text{(II.)}$$

which is the resultant of three wrenches of arbitrary intensity situated upon three given screws.* Introducing a linear vector function ϕ determined by the three conditions (Ex. 9, p. 103)

$$\mu_1 = \phi\lambda_1, \quad \mu_2 = \phi\lambda_2, \quad \mu_3 = \phi\lambda_3 \quad \text{(III.)}$$

we have $\mu = \phi\lambda$ if $\mu = \Sigma t_1\mu_1$ and $\lambda = \Sigma t_1\lambda_1$; ....(IV.)

and thus ($\phi\lambda$, λ), in which λ is arbitrary, is the *general expression for a wrench that can be compounded from wrenches on three given screws*, or conversely, that can be resolved into wrenches on the given screws.

To reduce the problem to its simplest form, let ϵ be the spin-vector of ϕ and let ϕ_0 be the self-conjugate part; then

$$\mu = V\epsilon\lambda + \phi_0\lambda = V\epsilon\lambda - aiSi\lambda - bjSj\lambda - ckSk\lambda \quad \text{(V.)}$$

where a, b and c are the roots of ϕ_0 and where i, j and k are the corresponding axes. Thus the wrench (μ, λ) may be compounded from the wrenches ($V\epsilon i + ai$, i), ($V\epsilon j + bj$, j), ($V\epsilon k + ck$, k), situated on screws whose axes i, j and k are mutually rectangular and which intersect at the extremity of the vector ϵ. The corresponding pitches are of course a, b and c; the latent roots of the self-conjugate part of the function ϕ.

The pitch of the wrench ($\phi\lambda$, λ) and the vector perpendicular on its axis are respectively (Art. 99)

$$p = S\phi\lambda \,.\, \lambda^{-1}, \quad \varpi = V\phi\lambda \,.\, \lambda^{-1}; \quad \text{(VI.)}$$

thus p is the reciprocal of the square of the radius of a quadric and the vector ϖ terminates on the surface represented by

$$S\frac{\psi'\varpi}{V\varpi\phi'\varpi} + 1 = 0, \quad \text{(VII.)}$$

because $V\varpi\phi'\varpi \parallel \lambda$ and therefore $\varpi = V\phi V\varpi\phi'\varpi(V\varpi\phi'\varpi)^{-1}$; and this surface is a quartic with three intersecting double lines—the axes of ϕ'. (Steiner's quartic surface.)

When the origin is taken at the extremity of the vector ϵ, the function ϕ is self-conjugate. This point is the *centre* of the *three-system* of screws. In terms of the pitch p and the vector η from the centre to any point on the axis of a screw of the system,

$$\mu = p\lambda + V\eta\lambda = \phi\lambda = \phi'\lambda, \quad \text{(VIII.)}$$

* See Joly, *Trans. R.I.A.*, Vol. xxx., Part xvi., and Vol. xxxii., Part viii.

so that λ is an axis and p the corresponding root of the function $\phi\rho - V\eta\rho$. The latent cubic of this function is (Art. 68, p. 98)

$$S\eta(\phi - p)\eta = m - pm' + p^2m'' - p^3; \quad \text{(IX.)}$$

and as η varies, this represents a quadric surface—one set of generators consisting of the axes of screws of given pitch which belong to the three-system. Three axes pass through an arbitrary point, and the sum of the corresponding pitches is constant and equal to the first invariant of ϕ. Two axes lie in an arbitrary plane $S\alpha\eta + 1 = 0$; their directions (compare (VI.)) are determined by

$$S\alpha\lambda = 0, \quad S\alpha\phi\lambda\lambda^{-1} + 1 = 0, \quad \text{(X.)}$$

and the corresponding pitches are the roots of

$$S\alpha(\psi - p\chi + p^2)\alpha = 1 \quad \text{(XI.)}$$

which is the condition that the plane should touch a quadric (IX.).

In order to reduce to a canonical form the two-system of wrenches compounded from two given wrenches (μ_1, λ_1) and (μ_2, λ_2), we assume in conformity with the foregoing a function ϕ which satisfies the relations

$$\phi\lambda_1 = \mu_1, \quad \phi\lambda_2 = \mu_2, \quad \phi V\lambda_1\lambda_2 = V\epsilon V\lambda_1\lambda_2 \quad \text{(XII.)}$$

where ϵ is the spin-vector of ϕ. The function $(\phi - V\epsilon)\rho$ will then be self-conjugate and will have a zero root, $V\lambda_1\lambda_2$ being the corresponding axis, and it will be expressible in the form $-aiSi\rho - bjSj\rho$. We have (Art. 27, p. 25)

$$\phi\rho = \mu_1 S\lambda_2(V\lambda_1\lambda_2)^{-1}\rho - \mu_2 S\lambda_1(V\lambda_1\lambda_2)^{-1}\rho$$
$$+ V\epsilon V\lambda_1\lambda_2 S(V\lambda_1\lambda_2)^{-1}\rho, \quad \text{(XIII.)}$$

and the spin-vector is deducible from the relation

$$2\epsilon = V\{(\mu_1\lambda_2 - \mu_2\lambda_1)(V\lambda_1\lambda_2)^{-1}\} + V\epsilon V\lambda_1\lambda_2(V\lambda_1\lambda_2)^{-1}.$$

Operating by $SV\lambda_1\lambda_2$ we find

$$2S\epsilon V\lambda_1\lambda_2 = S(\mu_1\lambda_2 - \mu_2\lambda_1)$$

which gives

$$\epsilon = V\{(\mu_1\lambda_2 - \mu_2\lambda_1)(V\lambda_1\lambda_2)^{-1}\} - \tfrac{1}{2}(V\lambda_1\lambda_2)^{-1}S(\mu_1\lambda_2 - \mu_2\lambda_1).$$

Taking the origin at the extremity of the vector ϵ, a wrench of unit intensity compounded from the two wrenches is determined by

$$\mu = \phi\lambda = ai\cos u + bj\sin u = p(i\cos u + j\sin u) + V\eta(i\cos u + j\sin u),$$
$$\lambda = i\cos u + j\sin u; \quad \text{(XIV.)}$$

whence the vector equation of the *cylindroid*—the locus of the central axes, and the equation for the pitch are

$$\eta = (b - a)k\sin u\cos u + t(i\cos u + j\sin u), \quad p = a\cos^2 u + b\sin^2 u$$

where u is the angle the axis makes with the vector i. The scalar equation of the cylindroid is found on elimination of u to be
$$\mathrm{TV}k\eta^2\mathrm{S}k\eta=(a-b)\mathrm{S}i\eta\mathrm{S}j\eta. \qquad \text{(XV.)}$$

To show that *in general a wrench may be resolved in one and only one way into components on six given screws,* or to reduce any pair of vectors μ and λ to the forms
$$\mu=\sum_1^6 t_1\mu_1, \quad \lambda=\sum_1^6 t_1\lambda_1, \qquad \text{(XVI.)}$$
where the vectors $\mu_1 \dots \mu_6$ and $\lambda_1 \dots \lambda_6$ are given, we assume in the first place
$$\mu_n=\phi_1\lambda_n, \quad (n=1, 2, 3); \quad \mu_n=\phi_2\lambda_n, \quad (n=4, 5, 6); \qquad \text{(XVII.)}$$
and writing $t_1\lambda_1+t_2\lambda_2+t_3\lambda_3=\tau_1, \quad t_4\lambda_4+t_5\lambda_5+t_6\lambda_6=\tau_2 \qquad$ (XVIII.)

we have
$$\mu=\phi_1\tau_1+\phi_2\tau_2, \quad \lambda=\tau_1+\tau_2;$$
or
$$\tau_1=(\phi_1-\phi_2)^{-1}(\mu-\phi_2\lambda), \quad \tau_2=(\phi_2-\phi_1)^{-1}(\mu-\phi_1\lambda). \qquad \text{(XIX.)}$$

Thus the vectors τ_1 and τ_2 are generally determinate and the scalars t follow from (XVIII).

Ex. 1. The locus of feet of perpendiculars from any point on the generators of a cylindroid is an ellipse.

[This is evident from the form of the equation (see Ex. 7, p. 64)
$$\varpi=\mathrm{V}(\mu_1+t\mu_2)(\lambda_1+t\lambda_2)^{-1}.]$$

Ex. 2. Find the locus of intersection of screws of the three-system $\mu=\phi\lambda$ whose axes are coplanar with the origin.

[If $\mu=\phi\lambda=p\lambda+\mathrm{V}\eta\lambda$, $\mu'=\phi\lambda'=p'\lambda+\mathrm{V}\eta\lambda'$ the axes intersect in η. Hence $(\phi-\mathrm{V}\eta-p)(\phi-\mathrm{V}\eta-p')$ destroys every vector coplanar with λ and λ' and in particular it destroys η if $\mathrm{S}\eta\lambda\lambda'=0$. Eliminating p and p' from $(\phi-\mathrm{V}\eta-p)(\phi-\mathrm{V}\eta-p')\eta=0$ we have the equation of the locus which may be written in the form
$$\mathrm{S}\frac{\psi\eta}{\mathrm{V}\eta\phi\eta}-1=0$$
which should be compared with (VII.).]

ART. **103**. To give an example of applying quaternions to a problem in statics, consider the case of a chain lying on a smooth surface and acted on by any force. Let ξ be the force per unit mass, ν the normal reaction per unit length, w the mass of the chain per unit length, and P the tension of the chain.

For equilibrium of an infinitesimal element at the extremity of ρ,
$$\mathrm{d}(P\mathrm{Ud}\rho)+w\xi\mathrm{Td}\rho+\nu\mathrm{Td}\rho=0, \quad \mathrm{S}\nu\mathrm{d}\rho=0, \qquad \text{(I.)}$$
the pull back at ρ being $-P\mathrm{Ud}\rho$ and the pull forward at $\rho+\mathrm{d}\rho$ being $+P\mathrm{Ud}\rho+\mathrm{d}(P\mathrm{Ud}\rho)$. When the length of the chain is

taken as the independent variable (Art. 85, p. 133), this may be written

$$P\,.\,\rho''+P'\rho'+w\xi+\nu=0,\quad \mathrm{S}\rho'\rho''=0,\quad \mathrm{S}\nu\rho'=0\,;\;\ldots\ldots\ldots(\text{II.})$$

and in virtue of the conditions it separates into

$$P\,.\,\rho''+w\rho'^{-1}\mathrm{V}\rho'\xi+\nu=0,\quad P'-w\mathrm{S}\xi\rho'=0,\;\ldots\ldots\ldots(\text{III.})$$

remembering that $\mathrm{T}\rho'=1$ so that $\mathrm{S}\xi\rho'^{-1}=-\mathrm{S}\xi\rho'$.

In certain cases the second of these equations can be integrated, and as it may be written

$$\mathrm{d}P-w\mathrm{S}\xi\mathrm{d}\rho=0\,;\quad P-\int w\mathrm{S}\xi\mathrm{d}\rho=const.=P_0\;\ldots\ldots\ldots(\text{IV.})$$

is the integral in question, P_0 being a constant.

The first equation gives the reaction

$$\mathrm{T}\nu^2=P^2\mathrm{T}\rho''^2-2Pw\mathrm{S}\xi\rho''+w^2\mathrm{T}(\mathrm{V}\xi\rho')^2\,;\;\ldots\ldots\ldots\ldots(\text{V.})$$

and shows that $P\rho''+w\rho'^{-1}\mathrm{V}\rho'\xi$ is normal to the surface, or that $P\rho'\rho''+w\mathrm{V}\rho'\xi$ is tangential. On elimination of the reaction ($\mathrm{T}\nu$),

$$P\mathrm{V}\rho''\mathrm{U}\nu+w\rho'\mathrm{S}\rho'^{-1}\xi\mathrm{U}\nu=0\,;\;\ldots\ldots\ldots\ldots\ldots\ldots(\text{VI.})$$

and the tension into the curvature into the cosine of the angle between the osculating and tangent planes is equal to the tangential component of the applied force per unit *length* which is at right angles to the tangent to the chain.

CHAPTER XII.

FINITE DISPLACEMENTS.

ART. **104**. To transfer a body from one position to another we may commence by rotating it until lines drawn in it receive their final directions. A translation without rotation which brings any point into its final position will complete the transference. In quaternions* if ϖ is the vector from a fixed point to any point in the body, the rotation changes the vectors to points in the body into $q\varpi q^{-1}$, and a translation τ added to this gives

$$\rho = \tau + q\varpi q^{-1} \quad \text{(I.)}$$

for the relation between vectors ϖ drawn to points in the initial position of the body, and vectors ρ drawn from the same origin to the same points in their final position.

This relation may be thrown into many various forms; for example

$$\rho = \tau' + q(\varpi - \epsilon)q^{-1}, \quad \tau' = \tau + q\epsilon q^{-1} \quad \text{(II.)}$$

shows that if the rotation were made about the extremity of the vector ϵ, the successive translation must be τ'; or we may first suppose a translation $(-\epsilon)$ effected, then the rotation about the origin and then the translation τ'.

Successive displacements are compounded according to the relations,

$$\rho = \tau' + q'\tau q'^{-1} + q'q\varpi q^{-1}q'^{-1} \quad \text{(III.)}$$

if

$$\rho' = \tau + q\varpi q^{-1}, \quad \rho = \tau' + q'\rho' q'^{-1};$$

and the order is all important for

$$\rho_{\prime} = \tau + q\tau' q^{-1} + qq'\varpi q'^{-1}q^{-1} \quad \text{(IV.)}$$

if

$$\rho_{\prime}' = \tau' + q'\varpi q'^{-1}, \quad \rho_{\prime} = \tau + q\rho_{\prime}' q^{-1};$$

and this vector $\rho_{\prime}$ is not equal to ρ. Even the rotations are different unless $qq' = q'q$, that is unless q and q' are coplanar; and the conditions that the order should be immaterial are

$$\mathrm{V}(\tau \mathrm{V}q') \,.\, q'^{-1} = \mathrm{V}(\tau' \mathrm{V}q) \,.\, q^{-1}; \quad qq' = q'q. \quad \text{(V.)}$$

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* The remarks in Art. 21 should be compared with this.

Small displacements are commutative in order of application. This is merely a particular case of a general theorem. Let any quantity a be changed by one operation into $a+f_1(a)$ where $f_1(a)$ is small, and into $a+f_2(a)$ by another operation, $f_2(a)$ being also small. Then to the second order of small quantities,

$$a+f_1(a)+f_2(a+f_1(a))=a+f_1(a)+f_2(a)$$
$$=a+f_2(a)+f_1(a+f_2(a)) \text{......(VI.)}$$

The simplest view of a displacement is as a *twist about a screw*, that is a rotation about a line coupled with a proportionate translation along the line. If η is the vector to any point on the line, and $P\mathrm{UV}q$ the translation along the line, we have to identify

$$\tau+q\varpi q^{-1}=\eta+P\mathrm{UV}q+q(\varpi-\eta)q^{-1}, \text{ ....(VII.)}$$

so that

$$\tau=\eta-q\eta q^{-1}+P\mathrm{UV}q=(\eta q-q\eta)q^{-1}+P\mathrm{UV}q$$
$$=2\mathrm{V}(\eta\mathrm{V}q)\,.\,q^{-1}+P\mathrm{UV}q,$$

and as it immediately appears that the first vector on the right is at right angles to $\mathrm{V}q$, we find on resolving τ along and perpendicular to $\mathrm{V}q$,

$$2\mathrm{V}\,.\,\eta\mathrm{V}q=\mathrm{V}\frac{\tau}{\mathrm{V}q}\,.\,\mathrm{V}q\,.\,q,\quad P=\mathrm{S}\frac{\tau}{\mathrm{UV}q}; \text{ ....(VIII.)}$$

and of these the first is the equation of the locus of the extremity of the vector η, or of the axis of the screw. The ratio of P to the angle of the rotation, or $P:2\angle q$, is the ratio of the pitch (p) to a whole revolution; and the pitch is therefore

$$p=\frac{\pi}{\angle q}\,.\,\mathrm{S}\frac{\tau}{\mathrm{UV}q}. \text{ ....(IX.)}$$

ART. **105**. Continuing to employ the same notation as in the last article, let us suppose that q and τ are functions of a variable parameter, the time t for example, and we shall have

$$\mathrm{d}\rho=\mathrm{d}\tau+\mathrm{V}\omega(\rho-\tau)\mathrm{d}t \quad \text{where} \quad \omega\mathrm{d}t=2\mathrm{V}\mathrm{d}qq^{-1},$$
$$\mathrm{d}\rho=\mathrm{d}\tau+q(\mathrm{V}\iota\varpi)q^{-1}\mathrm{d}t \quad \text{where} \quad \iota\mathrm{d}t=2\mathrm{V}q^{-1}\mathrm{d}q. \text{ ....(I.)}$$

To prove these relations observe that

$$\mathrm{d}\rho=\mathrm{d}\tau+\mathrm{d}q\,.\,\varpi q^{-1}+q\varpi\,.\,\mathrm{d}q^{-1}$$
$$=\mathrm{d}\tau+\mathrm{d}qq^{-1}\,.\,q\varpi q^{-1}-q\varpi q^{-1}\,.\,\mathrm{d}qq^{-1} \text{ ....(II.)}$$

remembering the expression for the differential of the reciprocal of a quaternion. This leads at once to the first relation since $p\lambda-\lambda p=2\mathrm{V}\,.\,\mathrm{V}p\lambda$ if p is any quaternion and λ any vector. The second relation is proved in quite an analogous manner.

The vector τ is the vector from the fixed origin of vectors ρ to the variable origin of vectors ϖ, and its derived with respect to the time is the velocity of that origin. The velocity of the extremity of the vector ρ is compounded of this velocity together with the velocity $V\omega(\rho-\tau)$ which is at right angles to ω and to $\rho-\tau$ and equal to the tensor of ω into the perpendicular from the extremity of $\rho-\tau$ on ω (the two vectors ω and $\rho-\tau$ being supposed to have a common origin). In fact the vector ω represents in magnitude and direction the angular velocity of the body.

Using fluxional notation for the velocities, we may write

$$\dot{\rho}=\dot{\tau}+V\omega(\rho-\tau)=\omega S\omega^{-1}\dot{\tau}+V\omega(\rho-\tau+V\omega^{-1}\dot{\tau}-x\omega), \ldots\text{(III.)}$$

thus analysing the instantaneous motion of the body into a rotation round a line coupled with a proportionate velocity of translation along the line; or, in Sir Robert Ball's phraseology, we have determined the instantaneous *twist-velocity* about the instantaneous screw; the expressions

$$\eta=\tau-V\omega^{-1}\dot{\tau}+x\omega, \quad p=S\omega^{-1}\dot{\tau} \ \ldots\ldots\ldots\ldots\text{(IV.)}$$

being the equation of the line or axis of instantaneous motion and the pitch of the instantaneous screw. (Compare Art. 99.)

When the equation of this axis is referred to the moving origin we may write it in the form

$$q^{-1}(\eta-\tau)q=-V\iota^{-1}q^{-1}\dot{\tau}q+x\iota=\eta' \ \text{ because } \ \omega=q\iota q^{-1}, \ldots\text{(V.)}$$

for $\omega=2V\dot{q}q^{-1}=2Vq(q^{-1}\dot{q})q^{-1}=2q(Vq^{-1}\dot{q})q^{-1}=q\iota q^{-1}$ by (I.). The line $\eta'=-V\iota^{-1}q^{-1}\dot{\tau}q+x\iota$ being supposed drawn in the body, the motion of the body brings it into coincidence at the proper instant with the instantaneous axis at the time t. Also the rotation converts ι into the angular velocity vector ω at the time t. Thus in dealing with the body itself it is convenient to use the vectors ι and ϖ, and in considering the motion of the body with regard to external objects, the vectors ω and ρ are preferably employed.

Let us no longer suppose the vector ϖ to be constant as in (II.). Then if the vectors ρ and ϖ are still connected by the first equation of the last article, we shall have instead of the first equation of the present article

$$\dot{\rho}=\dot{\tau}+V\omega(\rho-\tau)+q\dot{\varpi}q^{-1}, \quad \dot{\rho}=\dot{\tau}+q(V\iota\varpi+\dot{\varpi})q^{-1}; \ldots\text{(VI.)}$$

and more particularly when the vector τ is constantly zero,

$$\dot{\rho}=V\omega\rho+q\dot{\varpi}q^{-1}, \quad \dot{\rho}=q(\dot{\varpi}+V\iota\varpi)q^{-1}, \ \text{ if } \ \rho=q\varpi q^{-1}; \ldots\text{(VII.)}$$

and still more particularly

$$\dot{\omega}=q\dot{\iota}q^{-1} \ \text{ because } \ \omega=q\iota q^{-1}, \ V\omega\omega=0, \ V\iota\iota=0. \ldots\text{(VIII.)}$$

What we really do here is to compare the velocities of a point moving arbitrarily with respect to fixed objects and with respect to the moving body. The vector $\dot{\varpi}$ represents the velocity of the point relatively to the body, while $\dot{\rho}$ is its velocity relatively to fixed objects. Sometimes a notation such as

$$\frac{\partial(\rho)}{\partial t}=q\dot{\varpi}q^{-1};\quad \dot{\rho}=\mathrm{V}\omega\rho+\frac{\partial(\rho)}{\partial t};\quad \text{where}\quad \rho=q\varpi q^{-1}\ \ldots\ldots(\text{IX.})$$

may be employed—but it is not very explicit—to denote the variation of ρ arising from causes independent of the rotation; and in this notation we may replace (VIII.) by

$$\dot{\omega}=\frac{\partial(\omega)}{\partial t}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{X.})$$

which expresses that the rate of change of the angular velocity is independent of the rotation. We may for example suppose i, j and k to be fixed relatively to the vectors ϖ, and $\alpha=qiq^{-1}$, $\beta=qjq^{-1}$, $\gamma=qkq^{-1}$ to be unit vectors derived from these by the rotation. In this case if $\rho=\alpha x+\beta y+\gamma z$, the derived $\dot{\rho}$ takes account of the variations of α, β and γ as well as of x, y and z, while $\dfrac{\partial\rho}{\partial t}$ only refers to the variations of x, y and z and not at all to those of α, β and γ.

These results include the whole theory of fixed and moving axes, there being now no difficulty in writing down deriveds of any order. For example, on differentiating (VI.) again, we have

$$\ddot{\rho}=\ddot{\tau}+\mathrm{V}\dot{\omega}(\rho-\tau)+\mathrm{V}\omega(\dot{\rho}-\dot{\tau})+q\ddot{\varpi}q^{-1}+\mathrm{V}\omega q\dot{\varpi}q^{-1},$$

and on substituting for $\dot{\rho}$, the general formula of acceleration is

$$\ddot{\rho}=\ddot{\tau}+\mathrm{V}\dot{\omega}(\rho-\tau)+\mathrm{V}\omega\mathrm{V}\omega(\rho-\tau)+q\ddot{\varpi}q^{-1}+2\mathrm{V}\omega q\dot{\varpi}q^{-1},\ \ldots(\text{XI.})$$

which may of course be expressed in terms of ι.

In the case of a rigid body it is frequently convenient to replace (III.) by the relation

$$\dot{\rho}=\sigma+\mathrm{V}\omega\rho,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{XII.})$$

where σ is the velocity of the point of the body which instantaneously coincides with the fixed origin of vectors ρ. The acceleration of the point at the extremity of the vector ρ is

$$\ddot{\rho}=\dot{\sigma}+\mathrm{V}\omega\sigma+\mathrm{V}\dot{\omega}\rho+\mathrm{V}\omega\mathrm{V}\omega\rho,\ \ldots\ldots\ldots\ldots(\text{XIII.})$$

which follows on substitution for $\dot{\rho}$ in the result of differentiating (XII.).

As in Art. 102, we represent the *twist-velocity* of the body by the symbol (σ, ω), the fixed origin being taken as base-point, and we may replace (IV.) of the present article by

$$\sigma=(p+\mathrm{V}\eta)\omega;\quad p=\mathrm{S}\sigma\omega^{-1};\quad \eta=\mathrm{V}\sigma\omega^{-1}+x\omega.\ \ldots\ldots(\text{XIV.})$$

Ex. 1. The instantaneous twist-velocity of a body may be reduced to a pair of simultaneous angular velocities, β and β', round two lines, by means of the relations

$$\sigma = V\alpha\beta + V\alpha'\beta', \quad \omega = \beta + \beta',$$

where α and α' are vectors to points on the lines. [Compare Art. 102.]

Ex. 2. If p is the pitch of the instantaneous screw and if ω is the angular velocity of a rigid body, the velocity of any point in the body satisfies the relation

$$S\dot{\rho}\omega^{-1} = p\,;$$

and vectors drawn from a common origin to represent the simultaneous velocities of the points of the body terminate on a common plane.

Ex. 3. The locus of points having a velocity of given magnitude is a right circular cylinder

$$T\dot{\rho} = T(\sigma + V\sigma\rho) \quad \text{or} \quad TV\omega(\rho - \eta) = (T\dot{\rho}^2 - p^2T\omega^2)^{\frac{1}{2}},$$

coaxial with the instantaneous axis.

Ex. 4. Determine the acceleration centre of a body moving arbitrarily.

[In terms of σ and ω, if the acceleration of the point at the extremity of the vector α is instantaneously zero,

$$\dot{\sigma} + V\omega\sigma + V\dot{\omega}\alpha + \omega V\omega\alpha = 0 \quad \text{or} \quad \dot{\sigma} + V\omega\sigma + \phi\alpha = 0,$$

where $\phi\rho = V\dot{\omega}\rho + \omega V\omega\rho$. Hence $\psi\rho = -\dot{\omega}S\dot{\omega}\rho - V\,.\,\omega V\omega\dot{\omega}\,.\,\rho + \omega^3 S\omega\rho$ and the third invariant is $m = V\omega\dot{\omega}^2$, so that

$$\alpha V\omega\dot{\omega}^2 = (\dot{\omega}S\dot{\omega} + V\,.\,\omega V\omega\dot{\omega} - \omega^3 S\omega)\,.\,(\dot{\sigma} + V\omega\sigma).]$$

Ex. 5. The instantaneous acceleration of a point of a rigid body moving in any manner is a linear function of the vector to the point from the acceleration centre, or

$$\ddot{\rho} = \phi(\rho - \alpha) \quad \text{where} \quad \phi\rho = V\dot{\omega}\rho + V\omega V\omega\rho \quad \text{and} \quad \ddot{\alpha} = 0.$$

(a) The locus of points having instantaneous accelerations of given magnitude is one of a system of similar and coaxial ellipsoids

$$T\phi(\rho - \alpha) = T\ddot{\rho},$$

concentric with the acceleration centre, whose linear dimensions are proportional to the acceleration.

(b) The function ϕ is independent of the velocity of translation, and a change in that velocity merely alters the position of the acceleration centre and of the associated ellipsoids.

Ex. 6. The locus of points for which the magnitude of the velocity is momentarily constant is the quadric surface

$$S(\sigma + V\omega\rho)(\dot{\sigma} + V\dot{\omega}\rho) = 0 \quad \text{or} \quad S\{\dot{\alpha} + V\omega(\rho - \alpha)\}\phi(\rho - \alpha) = 0\,;$$

and the locus of points for which the direction of the velocity is momentarily constant is the twisted cubic

$$V(\sigma + V\omega\rho)(\dot{\sigma} + V\omega\sigma + V\dot{\omega}\rho + \omega V\omega\rho) = 0 \quad \text{or} \quad V\{\dot{\alpha} + V\omega(\rho - \alpha)\}\phi(\rho - \alpha) = 0.$$

(a) The equation of the twisted cubic may also be written in the form

$$\rho V\omega\dot{\omega}^2 = \{(\dot{\omega} - t\omega)S(\dot{\omega} - t\omega) + V\,.\,\omega V\omega\dot{\omega} - \omega^3 S\omega\}\,.\,(\dot{\sigma} - t\sigma + V\omega\sigma)$$

or

$$(\rho - \alpha)V\omega\dot{\omega}^2 = t\psi\dot{\alpha} + t^2(\omega S\dot{\omega}\dot{\alpha} + \dot{\omega}S\omega\dot{\alpha}) - t^3\omega S\omega\dot{\alpha},$$

where t is a variable scalar.

[For the twisted cubic we have $\phi\rho + \dot{\sigma} + V\omega\sigma = t(\sigma + V\omega\rho)$. Compare Ex. 4.]

ART. **106.** If the quaternion on which the rotation depends is a function of two variable parameters, u and v, we shall write

$$2\mathrm{V}\mathrm{d}qq^{-1}=\omega'\mathrm{d}u+\omega_{,}\mathrm{d}v,\quad \mathrm{d}q=\frac{\partial q}{\partial u}\mathrm{d}u+\frac{\partial q}{\partial v}\mathrm{d}u, \quad\text{(I.)}$$

and it must be observed that $\omega'\mathrm{d}u+\omega_{,}\mathrm{d}v$ is not a perfect differential. To determine the relation connecting ω' and $\omega_{,}$, suppose ϖ to be a constant vector and $\rho=q\varpi q^{-1}$. Then ρ is a function of u and v, and

$$\frac{\partial\rho}{\partial u}=\mathrm{V}\omega' q\varpi q^{-1},\quad \frac{\partial\rho}{\partial v}=\mathrm{V}\omega_{,}q\varpi q^{-1},\quad \frac{\partial^2\rho}{\partial u\partial v}=\frac{\partial^2\rho}{\partial v\partial u}. \quad\text{(II.)}$$

Calculating the second differentials,

$$\frac{\partial^2\rho}{\partial v\partial u}=\mathrm{V}\frac{\partial\omega'}{\partial v}q\varpi q^{-1}+\mathrm{V}\omega'\mathrm{V}\omega_{,}q\varpi q^{-1}=\mathrm{V}\frac{\partial\omega_{,}}{\partial u}q\varpi q^{-1}+\mathrm{V}\omega_{,}\mathrm{V}\omega' q\varpi q^{-1}=\frac{\partial^2\rho}{\partial u\partial v},$$

or, rearranging and observing that $\mathrm{V}\omega'\mathrm{V}\omega_{,}\lambda-\mathrm{V}\omega_{,}\mathrm{V}\omega'\lambda=\mathrm{V}\,.\,\mathrm{V}\omega'\omega_{,}\,.\,\lambda$, we have, because ϖ is an arbitrary vector,

$$\frac{\partial\omega'}{\partial v}-\frac{\partial\omega_{,}}{\partial u}+\mathrm{V}\omega'\omega_{,}=0. \quad\text{(III.)}$$

But again, by the last article and in the notation there explained,

$$\frac{\partial\omega'}{\partial v}=\mathrm{V}\omega_{,}\omega'+\frac{\partial(\omega')}{\partial v},\quad \frac{\partial\omega_{,}}{\partial u}=\mathrm{V}\omega'\omega_{,}+\frac{\partial(\omega_{,})}{\partial u}, \quad\text{(IV.)}$$

and accordingly we may replace (III.) by this new expression

$$\frac{\partial(\omega')}{\partial v}-\frac{\partial(\omega_{,})}{\partial u}-\mathrm{V}\omega'\omega_{,}=0. \quad\text{(V.)}$$

The results of this article have been employed in Art. 94 in connection with the theory of surfaces.

ART. **107.** In many investigations relating to rotations formulae of the type *

$$\rho=\gamma^z\beta^y\alpha^x\delta\alpha^{-x}\beta^{-y}\gamma^{-z} \quad\text{(I.)}$$

present themselves, and it may not be superfluous to make a few remarks about their reduction. It frequently happens that α, β and γ form a mutually rectangular unit system, and in this case if $\delta=a\alpha+b\beta+c\gamma$ we have

$$\rho=\gamma^z\beta^{2y}\gamma^z\,.\,a\alpha+\gamma^z\beta^y\alpha^{2x}\beta^{-y}\gamma^z\,.\,b\beta+\gamma^z\beta^y\alpha^{2x}\beta^y\gamma^{-z}\,.\,c\gamma, \quad\text{(II.)}$$

when we apply the general relation

$$\alpha^x\beta=\beta\alpha^{-x}\ \text{ if }\ \mathrm{S}\alpha\beta=0,\ \ \mathrm{T}\alpha=1. \quad\text{(III.)}$$

In order to reduce the coefficient of $b\beta$ for instance, it is generally best to start from the *central* term, α^{2x} in this case, and to replace it by $\cos\pi x+\alpha\sin\pi x$, and similarly for successive reductions. Thus we avoid introducing the sines and cosines of the halves of the angles of rotation.

It is worth while noticing that

$$\mathrm{d}\alpha^x\,.\,\alpha^{-x}=\frac{\pi}{2}\mathrm{d}x\,.\,\alpha+\tfrac{1}{2}\mathrm{d}\alpha(\alpha^{1-2x}+\alpha^{-1}) \quad\text{(IV.)}$$

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* It may be advisable to refer again to Chap. IV. and its examples.

is expressible in terms of the whole angle using the relation

$$S\alpha^{x-1}.\alpha^{-x}=\tfrac{1}{2}(1+K)\alpha^{x-1}.\alpha^{-x}.$$

The general relation connecting two quaternions p and q and two scalars x and y,

$$(p^x q p^{-x})^y = p^x q^y p^{-x}, \quad \text{(v.)}$$

will often be found useful.

Ex. 1. A planet rotates about its axis γ in the period $2m^{-1}$ and a satellite describes a circular orbit round the planet in the period $2n^{-1}$; show that the motion relative to the planet is represented by

$$\rho=(\gamma^{-mt}\delta\gamma^{mt})^{nt}.\gamma^{-mt}\epsilon\gamma^{mt}.(\gamma^{-mt}\delta\gamma^{mt})^{-nt}, \quad S\delta\epsilon=0,$$

the vectors in this expression being all fixed relatively to the planet; and reduce the equation to

$$\rho=\gamma^{-mt}\delta^{nt}\epsilon\delta^{-nt}\gamma^{mt}.$$

(*a*) By taking the epoch when the satellite is in the plane of the equator, the equation may be simplified to

$$\rho=r\gamma^{-mt}\beta^{a}\gamma^{nt}\beta.\gamma^{-nt}\beta^{-a}\gamma^{mt}, \quad S\beta\gamma=0$$

where r is the radius of the orbit and where πa is the angle between the plane of the orbit and the equator.

(*b*) The equation may also be written

$$\begin{aligned}\rho=&\,r\alpha(\cos\pi nt\sin\pi mt-\cos\pi a\sin\pi nt\cos\pi mt)\\&+r\beta(\cos\pi nt\cos\pi mt+\cos\pi a\sin\pi nt\sin\pi mt)\\&+r\gamma\sin\pi a\sin\pi nt\end{aligned}$$

where $\alpha=\beta\gamma.$

(*c*) The condition for a stationary point may be written in the form

$$mV\gamma\beta^{a}\gamma^{nt}\beta\gamma^{-nt}\beta^{-a}+n\beta^{a}\gamma^{nt}\alpha\gamma^{-nt}\beta^{-a}=0,$$

or

$$n\alpha+mV\gamma^{-nt}\beta^{-a}\gamma\beta^{a}\gamma^{nt}\beta=0,$$

and this is equivalent to

$$n=m\cos\pi a, \quad \cos\pi nt=0.$$

Ex. 2. Unit vectors α, β and γ are directed respectively to the point of upper culmination on the celestial equator, to the east point and to the north celestial pole, while i, j and k are directed to the south point, the east point and the zenith. Show that the vector directed to a star may be expressed in the forms

$$\sigma=\gamma^{-z}\beta^{-y}\alpha\beta^{y}\gamma^{z}=k^{-w}j^{-v}ij^{v}k^{w}$$

where πz is the hour-angle west, πy the declination, πw the azimuth west, and πv the altitude.

(*a*) If πb is the latitude of the place of observation, show that

$$k=\beta^{-b}\alpha\beta^{b};$$

and obtain the quaternion equation

$$\gamma^{z}\beta^{2y}\gamma^{z}=\beta^{b}\alpha^{-w}\beta^{2v-1}\alpha^{w}\beta^{b},$$

and hence deduce the formulae of transformation from one set of coordinates to the other.

Ex. 3. Assuming the effect of refraction to be K times the tangent of the zenith distance, prove that the vector to the apparent place of a star is $\sigma+\varpi$ where

$$\varpi=K.\frac{\mathrm{V}k\sigma\,.\,\sigma}{\mathrm{S}k\sigma}=K.\gamma^{-z}\beta^{-y}.\frac{\mathrm{V}\beta^{y}\gamma^{z}k\gamma^{-z}\beta^{-y}\alpha\,.\,\alpha}{\mathrm{S}\beta^{y}\gamma^{z}k\gamma^{-z}\beta^{-y}\alpha}.\beta^{y}\gamma^{z}.$$

(a) Substituting for k in terms of α and β (Ex. 2 (a)), verify the successive steps of the transformation

$$\beta^{y}\gamma^{z}k\gamma^{-z}\beta^{-y}\alpha=-\beta^{y}\gamma^{z}\beta^{-2b}\gamma^{z}\beta^{y}=-\beta^{y}\gamma^{2z}\beta^{y}\cos\pi b+\beta^{2y+1}\sin\pi b$$
$$=-\sin\pi b\sin\pi y-\cos\pi b\cos\pi y\cos\pi z+\beta(\sin\pi b\cos\pi y-\cos\pi b\sin\pi y\cos\pi z)$$
$$-\gamma\cos\pi b\sin\pi z.$$

(b) Show that the expression for ϖ reduces to the form

$$\varpi=\frac{\beta'\cos\pi b\sin\pi z+\gamma'(\sin\pi b\cos\pi y-\cos\pi b\sin\pi y\cos\pi z)}{\sin\pi b\sin\pi y+\cos\pi b\cos\pi y\cos\pi z},$$

where β' and γ' are unit vectors tangential respectively to the parallel of declination and to the circle of declination.

(c) If q is the parallactic angle and ζ the zenith distance, show that

$$\sigma^{-\frac{1}{2}\pi q}=\mathrm{U}\frac{\mathrm{V}k\sigma}{\mathrm{V}\gamma\sigma},\quad \varpi=K\sigma^{1-\frac{1}{2}\pi q}.\mathrm{UV}\gamma\sigma.\tan\zeta.$$

Ex. 4. An equatorial telescope in imperfect adjustment is directed to a star, and the circle readings are observed to be $(y+y')\pi$ and $(z+z')\pi$ where y' and z' are small; if for zero circle readings the direction of the telescope is $\alpha+\alpha'$, that of the declination axis $\beta+\beta'$ and that of the polar axis $\gamma+\gamma'$ where α', β' and γ' are small vectors perpendicular respectively to α, β and γ, show that

$$\sigma=(\gamma+\gamma')^{-(z+z')}(\beta+\beta')^{-(y+y')}(\alpha+\alpha')(\beta+\beta')^{y+y'}(\gamma+\gamma')^{z+z'};$$

and neglecting small terms of the second order obtain the relation

$$\mathrm{V}\beta^{y}\{\pi z'\gamma+(\gamma^{2z-1}+\gamma)\gamma'\}\beta^{-y}\alpha+\mathrm{V}\{\pi y'\beta+(\beta^{2y-1}+\beta)\beta'\}\alpha=\alpha'.$$

From this and two similar equations corresponding to the results of setting the telescope on two other known stars, deduce the errors in the adjustment which are represented by the small vectors α', β' and γ'.

Ex. 5. The unit of length is taken equal to the focal length of a photographic telescope in perfect adjustment so that were it not for refraction the image of a star would remain fixed on the photographic plate. Assuming the effect of refraction to be K times the tangent of the zenith distance, show that the image describes on the plate a curve represented by

$$\varpi=-K\frac{\mathrm{V}\gamma^{z}\kappa\gamma^{-z}\sigma\,.\,\sigma}{\mathrm{S}\gamma^{z}\kappa\gamma^{-z}\sigma}$$

where $z\pi$ is the hour angle reckoned towards the west, and where σ, γ and κ are three (coplanar) unit vectors fixed relatively to the plate and directed respectively to the star, to the north celestial pole and, when the telescope is on the star in the meridian, to the zenith.

(a) Prove that this curve represents a conic, or a portion of a conic, and that it is the intersection of the plane and cone

$$\mathrm{S}\varpi\sigma=0,\quad \mathrm{S}\gamma\mathrm{U}(\sigma K-\varpi)=\mathrm{S}\gamma\kappa,$$

and consider the arrangement of the curves for various values of K and for stars of different declinations.

Ex. 6. The positions of stars are determined by taking transits with a telescope movable about a fixed axis. Show that the hour-angle πz at the time of transit and the declination πy are connected with the reading πu of a circle fixed to the telescope at right angles to the fixed axis by the quaternion equation

$$\gamma^z \beta^{2y} \gamma^z = \gamma^c \beta^b \alpha^{-u} \beta^{-b} \gamma^{c'} \beta^{2b'} \gamma^{c'} \beta^{-b} \alpha^u \beta^b \gamma^c$$

where b, b', c and c' are constants of the instrument, α, β and γ having the same signification as in Ex. 2.

(a) If δ is a unit vector along the axis round which the telescope turns the equation may be written in the form

$$\gamma^z \beta^{2y} \gamma^z = \alpha^{-1} \delta \mathrm{S} \epsilon \delta^{-1} + \alpha^{-1} \mathrm{V} \epsilon \delta^{-1} . \delta^{2u+1};$$

and for an *almucantar* whose line of collimation makes a constant angle (πa) with the vertical and is in the meridian when $u=0$, the equation is

$$\gamma^z \beta^{2y} \gamma^z = \beta^{2b} \cos \pi a + \beta^{b-1} \alpha^{2u} \beta^b \sin \pi a$$

where πb is the latitude of the place.

Ex. 7. If $\mathrm{U}\sigma$ is the unit vector towards the centre of a planet; $\mathrm{U}\sigma + \tau$ the vector towards a marking on the planet in latitude l; γ the unit vector along the planet's axis of rotation; α the unit vector from the planet's centre towards the point on its equator on the meridian through the marking; if P is the time of rotation of the planet on its axis and s the angular semi-diameter at the time of observation, show that

$$\gamma \sin l + \gamma^{4tP^{-1}} \alpha \cos l = \tau s^{-1} - \mathrm{U}\sigma (1 + \tau^2 s^{-2})^{\frac{1}{2}},$$

where t is the time of observation measured from some selected epoch.

(a) Denoting the vector on the right by η, show that η terminates on a fixed circle and verify that

$$\gamma \operatorname{cosec} l = -\mathrm{V}(\eta_2 \eta_3 + \eta_3 \eta_1 + \eta_1 \eta_2)(\mathrm{S} \eta_1 \eta_2 \eta_3)^{-1}$$

where η_1, η_2 and η_3 are the values of the vector η at three times of observation.

(c) Show how to deduce the time of rotation.

Ex. 8. A polar axis having a fixed direction γ carries a declination axis initially parallel to β on which is mounted a telescope initially parallel to α. The vectors being all of unit length and the instrument being completely out of adjustment so that no conditions of rectangularity are even approximately satisfied, show that when the direction of the telescope is changed to α' by a rotation round the declination axis followed by a rotation round the polar axis,

$$\alpha' = \gamma^z \beta^y \alpha \beta^{-y} \gamma^{-z},$$

while if the rotation is first made round the polar axis and then round the declination axis,

$$\alpha' = (\gamma^z \beta \gamma^{-z})^y \gamma^z \alpha \gamma^{-z} (\gamma^z \beta \gamma^{-z})^y,$$

and prove the equivalence of these two expressions.

(a) If u and v are the tangents of half the angles of rotation round the polar axis and the declination axis respectively, show that the vector equation

$$\alpha - \alpha' + u \mathrm{V} \gamma (\alpha + \alpha') + v \mathrm{V} \beta (\alpha + \alpha') + uv \{(\alpha - \alpha') \mathrm{S} \beta \gamma + \mathrm{V} . \mathrm{V} \gamma \beta (\alpha + \alpha')\} = 0$$

serves to determine both u and v.

(b) Deduce from this the scalar quadratic equation in u:

$$\mathrm{S} \beta (\alpha - \alpha') - 2u \mathrm{S} \gamma \beta \alpha' - u^2 \mathrm{S} \gamma (\alpha - \alpha') \mathrm{S} \gamma \beta - u^2 \mathrm{S} \gamma \mathrm{V} \gamma \beta (\alpha + \alpha') = 0.$$

CHAPTER XIII.

STRAIN.

ART. **108.** Homogeneous strain converts vectors (ρ) in an unstrained body into vectors ($\sigma = \phi\rho$) in the manner described in the chapter on the linear vector function (Arts. 63, 64), but the transformation is of less generality. The order of rotation from $\phi\alpha$ to $\phi\beta$ to $\phi\gamma$ must agree with that from α to β to γ in the case of a physical strain, for otherwise a positive volume would be converted into a negative volume (Art. 24). In other words the third invariant of the function ϕ must be positive, or the condition

$$m > 0 \text{ ....(I.)}$$

must be satisfied. This requires one latent root of ϕ to be real and positive, and when the roots are all real this is obviously the case. When two of the roots are imaginary, $g' + \sqrt{-1}g''$ and $g' - \sqrt{-1}g''$, the third invariant is $(g'^2 + g''^2)g$ where g is the remaining latent root; so that here again one root is positive. It follows from this that *in every homogeneous strain one direction at least remains unchanged*, for we have

$$\mathrm{U}\phi\alpha = \mathrm{U}\alpha \text{ if } \phi\alpha = g\alpha, \; g > 0. \text{ ....(II.)}$$

If the three latent roots are positive, three lines remain unrotated. In the case of a *pure strain* three mutually rectangular directions remain unchanged, and the function ϕ is *self-conjugate with positive latent roots*. The decomposition of a linear function into a self-conjugate function preceded or followed by a rotation has been considered in Art. 70; and by selecting the square root $(\phi\phi')^{\frac{1}{2}}$ of the function $\phi\phi'$ which has all its latent roots positive we decompose, without ambiguity, an arbitrary strain into a rotation followed by a pure strain.

A sphere $\mathrm{T}\rho = r$ is converted into an ellipsoid—the *strain ellipsoid*,*

$$\mathrm{T}\phi^{-1}\sigma = r \text{ or } \mathrm{S}\sigma\phi'^{-1}\phi^{-1}\sigma + r^2 = 0; \text{ ....(III.)}$$

* The results of Art. 70 show that the surface is ellipsoidal.

and the axes of this surface are parallel to the axes of $\phi'^{-1}\phi^{-1}$ or of its inverse $\phi\phi'$ (not $\phi'\phi$). And the ellipsoid

$$T\phi\rho = r \quad \text{or} \quad S\rho\phi'\phi\rho + r^2 = 0 \quad\text{(IV.)}$$

is converted into the sphere $T\sigma = r$. The rôle of the functions $\phi\phi'$ and $\phi'\phi$ is quite analogous to that of the functions of Art. 101, p. 161, denoted by the same symbols.

ART. **109**. A shear is represented by the function

$$\phi\rho = \rho - \beta S\alpha\rho \quad \text{where} \quad S\alpha\beta = 0, \quad\text{(I.)}$$

for a point in the body is displaced parallel to a fixed direction $(U\beta)$ through a distance proportional to its distance from a plane $(S\alpha\rho = 0)$ parallel to the fixed direction $(U\beta)$. In all cases the *displacement* of a point—the extremity of the vector ρ—is $\phi\rho - \rho$. A shear accompanied by a uniform dilatation is represented by

$$\phi\rho = g\rho - \beta S\alpha\rho, \quad S\alpha\beta = 0, \quad\text{(II.)}$$

the ratio of the changed volume to the original being that of g^3 to unity.

The function $\phi\rho = gq\rho q^{-1} - q\beta q^{-1}S\alpha\rho,\ S\alpha\beta = 0,$ ....(III.)

represents a dilatation and a shear followed by a rotation, and this function involves eight constants—three in Uq, one in g, three in $\alpha T\beta$ and one in $U\beta$ (because $S\alpha\beta = 0$)—just one less than in the general function.

Omitting the condition $S\alpha\beta = 0$ in (III.), the function involves nine constants, and the function

$$\phi\rho = gq\rho q^{-1} - q\beta q^{-1}S\alpha\rho \quad\text{(IV.)}$$

is capable of representing the *most general strain which may be produced by shifting in a fixed direction* $(U\beta)$ *planes parallel to a fixed plane* $(S\alpha\rho = 0)$ *by an amount* $(-g^{-1}\beta S\alpha\rho)$ *proportional to the perpendicular distance from the fixed plane; by altering all lines in the ratio* g *to unity, and by superposing a rotation.* To prove this we identify

$$\begin{aligned}\phi'\phi\rho &= (g - \alpha S\beta)(g - \beta S\alpha)\rho \\ &= g^2\rho - \alpha S(g\beta - \tfrac{1}{2}\beta^2\alpha)\rho - (g\beta - \tfrac{1}{2}\beta^2\alpha)S\alpha\rho \quad\text{(V.)}\end{aligned}$$

with Hamilton's cyclic form (Art. 77) for the general self-conjugate ellipsoidal function so that the third invariant of ϕ may be positive or that

$$g^2(g - S\alpha\beta) > 0; \quad\text{(VI.)}$$

in other words we suppose ϕ to be a given function, and it is required to determine α, β, g and q. If a^2, b^2, c^2 are the latent roots of the general self-conjugate function

$$\phi'\phi\rho = b^2\rho + \lambda S\mu\rho + \mu S\lambda\rho, \quad\text{(VII.)}$$

$$2S\lambda\mu = a^2 + c^2 - 2b^2, \quad 2T\lambda\mu = a^2 - c^2$$

(compare Art. 77 (II.) and Ex. 2), we have on comparison with (V.)

$$g=b,\ a=-\lambda,\ b\beta=\mu+\tfrac{1}{2}\lambda T\beta^2, \ \ldots\ldots\ldots\ldots\ldots\text{(VIII.)}$$

whence substituting from (VII.) in $bT\beta=T(\mu+\frac{1}{2}\lambda T\beta^2)$, we find the quadratic in $T\beta^2$,

$$T\lambda^4 T\beta^4-2(a^2+c^2)T\lambda^2 T\beta^2+(a^2-c^2)^2=0, \ \ldots\ldots\ldots\text{(IX.)}$$

whose roots are $T\lambda^2 T\beta^2=(a\pm c)^2$. These give

$$b\beta=\mu-\tfrac{1}{2}\lambda^{-1}(a\pm c)^2,$$

and it follows from (VI.) that we must select the negative sign. Thus we have definitely by (VIII.)

$$g=b,\ a=-\lambda,\ b\beta=\mu-\tfrac{1}{2}\lambda^{-1}(a-c)^2; \ \ldots\ldots\ldots\ldots\text{(X.)}$$

and the rotation may be determined as in Art. 70. A second solution is obtained by interchanging λ and μ.

Ex. 1. Prove that the necessary and sufficient conditions that the function ϕ should represent a uniform dilatation and a dilatation accompanying a shear, are respectively

$$\phi-g=0,\quad (\phi-g)^2=0.$$

[These are excellent examples of the degradation of the symbolic cubic. Art. 66, p. 95.]

Ex. 2. If the function ϕ represents a uniform dilatation and two superposed shears,

$$m''m^{\frac{1}{3}}=m'.$$

[Assuming $\phi\rho=g(1-\beta' S\alpha')(1-\beta S\alpha)\rho$, $S\alpha\beta=S\alpha'\beta'=0$, it is necessary to prove that g is a root of ϕ, and that it is equal to the cube root of m. It may be shown that the converse is also true.]

Ex. 3. The strain produced by two successive pure strains is generally impure.

[Two functions are commutative in order of operation only if they are coaxial (Art. 66, Ex. 2, p. 95).]

ART. **110.** Lines in the unstrained body whose lengths are altered in a given ratio g are parallel to edges of the quadric cone

$$T\phi U\rho=g,\ \text{ or }\ SU\rho(\phi'\phi-g^2)U\rho=0 \ \ldots\ldots\ldots\ldots\ldots\text{(I.)}$$

—one of a concyclic system; and by (VII.) this equation may be replaced by

$$2S\lambda U\rho S\mu U\rho=b^2-g^2,\ \text{ or }\ \sin u\sin v=(b^2-g^2)(a^2-c^2)^{-1}, \ \ldots\text{(II.)}$$

where u and v are the angles a line makes with the cyclic planes of the function $\phi'\phi$. The ratio g for any direction is the reciprocal of the parallel radius of the quadric (compare Art. 108 (IV.)),

$$T\phi\rho=1. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(III.)}$$

If the inclination of the vector β to α remains unchanged, the condition

$$\mathrm{SU}\,.\,\alpha\beta=\mathrm{SU}\,.\,\phi\alpha\phi\beta,\ \text{or}\ \mathrm{S}\alpha\beta\,.\,\mathrm{T}\phi\alpha\phi\beta=\mathrm{S}\alpha\phi'\phi\beta\,.\,\mathrm{T}\alpha\beta\ \ldots(\text{IV.})$$

is satisfied, and the locus of the vectors β is the quartic cone

$$\mathrm{S}\alpha\beta^2\mathrm{S}\alpha\phi'\phi\alpha\mathrm{S}\beta\phi'\phi\beta=\mathrm{S}\alpha\phi'\phi\beta^2\,.\,\alpha^2\beta^2,\ \ldots\ldots\ldots\ldots(\text{V.})$$

which has α and $\mathrm{V}\alpha\phi'\phi\alpha$ for double edges. Substituting $\alpha+t\alpha\lambda$ for β in this equation, we get for the edges in the plane $\mathrm{S}\lambda\rho=0$, which passes through α,

$$\beta=\mathrm{V}\lambda(\phi'\phi\alpha\pm\alpha(\mathrm{S}\lambda^{-1}\psi\psi'\lambda)^{\frac{1}{2}}),\ \ldots\ldots\ldots\ldots\ldots\ldots(\text{VI.})$$

after discarding the factor t^2. These edges are real for all directions of the vector λ, and it easily appears that the upper sign corresponds to $\mathrm{SU}\,.\,\alpha\beta=+\mathrm{SU}\,.\,\phi\alpha\phi\beta$, while the lower sign corresponds to $\mathrm{SU}\,.\,\alpha\beta=-\mathrm{SU}\,.\,\phi\alpha\phi\beta$ on comparing the signs of $\mathrm{S}\alpha\beta$ and $\mathrm{S}\beta\phi'\phi\alpha$. The lower sign corresponds to the case in which the angle between $\phi\alpha$ and $\phi\beta$ is the supplement of that between α and β. The vector $\mathrm{V}\alpha\phi'\phi\alpha$ alone remains at right angles to α, and (Art. 75 (IV.)) this vector is parallel to the second principal axis of the section of (III.) of which α is a principal axis.

If an arbitrary rotation is superposed on the strain, the cone (IV.) is the locus of lines which together with α can be unrotated lines—or axes of $q(\phi\rho)q^{-1}$. The latent root corresponding to any edge (β) is (compare (I.)) $\pm\mathrm{T}\phi\mathrm{U}\beta$. To determine the rotation which must be superposed on the strain so as to leave unrotated two vectors α and β satisfying the condition (IV.) we may utilize Ex. 6, Chapter III., p. 26, and find the rotation which converts $\mathrm{U}\phi\alpha$ and $\mathrm{U}\phi\beta$ into $\pm\mathrm{U}\alpha$ and $\pm\mathrm{U}\beta$, having as in (VI.) due regard to the indeterminate sign. It is possible to superpose a rotation on a strain so that all the lines in a plane may be unrotated. It is only necessary to reduce the function ϕ to the form given in Art. 109 (IV.), and we have

$$q^{-1}\,.\,\phi\rho\,.\,q=g\rho-\beta\mathrm{S}\alpha\rho,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{VII.})$$

and the lines in the plane $\mathrm{S}\alpha\rho=0$ (or $\mathrm{S}\lambda\rho=0$, compare Art. 109 (X.))—a cyclic plane of $\phi'\phi$—are unrotated.

ART. **111**. The *displacement* at the extremity of the vector ρ produced by the strain is

$$\delta=\sigma-\rho=(\phi-1)\rho=\rho(\mathrm{S}\rho^{-1}\phi\rho-1)+\rho\mathrm{V}\rho^{-1}\phi\rho,\ \ldots\ldots(\text{I.})$$

which we have resolved along and at right angles to the vector ρ. When unity is a latent root of the function ϕ, the displacement is parallel to a fixed plane—that of the axes of ϕ complementary to the unstrained and unrotated axis corresponding to the root unity. (See Art. 66 (X.), p. 94.)

In general, provided the greatest and least roots of $\phi'\phi$ are greater and less than unity, it is possible by the last article to superpose a rotation on the strain so that the resulting displacement may be everywhere parallel to a fixed plane.

The quantity $$e = S\rho^{-1}(\phi - 1)\rho \quad\text{(II.)}$$
is called the *elongation*, and it is numerically equal to the reciprocal of the square of the radius of the *elongation quadric*

$$S\rho(\phi_0 - 1)\rho = -1, \quad (\phi_0 = \tfrac{1}{2}(\phi + \phi')), \quad\text{(III.)}$$

which is parallel to the vector ρ. This quadric may be an ellipsoid or a hyperboloid according to the relative magnitudes of the roots of ϕ_0 and unity.

The component of the displacement perpendicular to ρ may be written in the form

$$V\eta\rho = V\phi\rho\rho^{-1} \,.\, \rho = V\epsilon\rho + V\phi_0\rho\rho^{-1} \,.\, \rho \quad\text{(IV.)}$$

where ϵ is the spin-vector of ϕ, and (Art. 75 (IV.)) the vector $V\phi_0\rho\rho^{-1}$ is parallel to the second principal axis of the section of (III.) of which ρ is a principal axis. The magnitude of this vector ($TV\rho^{-1}(\phi_0 - 1)\rho$) is numerically equal to the area of the triangle formed by lines drawn along $U\rho$ and along the central perpendicular on the corresponding tangent plane of the elongation quadric—the lengths of these lines being the reciprocals of those of the central radius and the central perpendicular.

ART. **112.** When the strain is not homogeneous, if the point P is strained to Q, the relation between the vectors $\rho(=\text{OP})$ and $\sigma(=\text{OQ})$ ceases to be linear, but we always have the corresponding differentials linearly related, or

$$d\sigma = \phi d\rho \quad\text{if}\quad \sigma = \theta(\rho), \quad\text{(I.)}$$

θ being any function of ρ, and $\phi d\rho$ being a linear function of $d\rho$ involving the vector ρ in its constitution. So long then as we confine our attention to the limits of vanishing and corresponding regions at Q and P, so that the vector ρ does not vary, the treatment of this general case is precisely the same as in the case of homogeneous strain.

In terms of the operator ∇, $d\bar{\sigma} =$

$$d\sigma = -Sd\rho\nabla \,.\, \sigma, \quad\text{(II.)}$$

so that if α is any vector which is not subject to the operation of ∇,

$$\phi\alpha = -S\alpha\nabla \,.\, \sigma, \text{ and } \phi'\alpha = -\nabla S\alpha\sigma, \quad\text{(III.)}$$ $\phi'\bar{\alpha} = \nabla$

as we may verify in many ways* by the results of Arts. 56 and 57; and in the same way it is not hard to see that we may write

$$(\phi - \phi')\alpha = V \,.\, V\nabla\sigma \,.\, \alpha, \quad V\nabla\sigma = 2\epsilon, \quad\text{(IV.)}$$

* For example $\phi\alpha = +\Sigma S\alpha V\mu\nu \,.\, \dfrac{\partial\sigma}{\partial u} : S\lambda\mu\nu$; $\phi'\alpha = \Sigma V\mu\nu S\alpha\dfrac{\partial\sigma}{\partial u} : S\lambda\mu\nu$.

where ϵ is the spin-vector of a. Thus for a pure strain at all points, we must have

$$\mathrm{V}\nabla\sigma = 0, \text{ or } \sigma = \nabla P \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(V.)}$$

(Art. 56) where P is a scalar function of ρ. (See p. 74.)

ART. **113**. For small strains it is convenient to change the notation and to consider the displacement of a point produced by the strain rather than the relation between the vectors to the strained and unstrained positions of the point. We write therefore for a homogeneous small strain

$$\sigma = \rho + \phi\rho, \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I.)}$$

replacing the function ϕ of earlier articles by $1+\phi$, the function ϕ being now small, or $\mathrm{T}\phi\rho$ being small in comparison with $\mathrm{T}\rho$. Apart from its smallness, however, the new function is of a more general character than the old. We may for example have the order of rotation from ϕa to $\phi\beta$ to $\phi\gamma$ different from that from a to β to γ without violating the physical reality of the strain. In fact the ratio of volumes is now

$$lim. \frac{\mathrm{S}(a+\phi a)(\beta+\phi\beta)(\gamma+\phi\gamma)}{\mathrm{S}a\beta\gamma} = \frac{\mathrm{S}a\beta\gamma + \Sigma\mathrm{S}\phi a\beta\gamma}{\mathrm{S}a\beta\gamma} = 1 + m'', \;\ldots\text{(II.)}$$

and m'' is small in comparison with unity.

Small strains are superposable (cf. Art. 104 (VI.), p. 169), or

$$(1+\phi_1)(1+\phi_2)\rho = (1+\phi_1+\phi_2)\rho = (1+\phi_2)(1+\phi_1)\rho, \;\ldots\text{(III.)}$$

because we agree to neglect the terms of the second order $\phi_1\phi_2\rho$ and $\phi_2\phi_1\rho$.

A small strain is resolvable into a pure strain and a small rotation by the relation

$$\rho + \phi\rho = \rho + \phi_0\rho + \mathrm{V}\epsilon\rho = (1+\mathrm{V}\epsilon)(1+\phi_0)\rho = (1+\phi_0)(1+\mathrm{V}\epsilon)\rho \quad \text{(IV.)}$$

where ϕ is the self-conjugate part of ϕ and where ϵ is its spin-vector.

We may write

$$\rho + \mathrm{V}\epsilon\rho = (1+\tfrac{1}{2}\epsilon)\rho(1+\tfrac{1}{2}\epsilon)^{-1} = \rho + \tfrac{1}{2}\epsilon\rho - \tfrac{1}{2}\rho\epsilon. \;\ldots\ldots\ldots\text{(V.)}$$

The strain quadric now becomes

$$\sigma^2 - 2\mathrm{S}\sigma\phi_0\sigma + r^2 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(VI.)}$$

if $\rho^2 + r^2 = 0$; for $\rho = (1-\phi)\sigma$ if $\sigma = (1+\phi)\rho$, since approximately $\rho = (1-\phi^2)\rho = (1-\phi)\sigma$.

For non-homogeneous small strains, suppose $\theta(\rho)$ to be the displacement of the extremity of the vector ρ. Equation (I.) then becomes

$$\sigma = \rho + \theta(\rho), \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(VII.)}$$

and for a neighbouring point

$$\mathrm{d}\sigma = \mathrm{d}\rho + \phi\mathrm{d}\rho = \mathrm{d}\rho - \mathrm{S}\mathrm{d}\rho\nabla\,.\,\theta\rho. \;\ldots\ldots\ldots\ldots\ldots\text{(VIII.)}$$

Confining the attention to points in the neighbourhood of the extremity of ρ, equation (VIII.) is of the same form as (I.), and the results of the present article apply if we regard the function ϕ already employed as having the meaning assigned to the same symbol in (VIII.), and if we suppose that the vectors ρ throughout the article are small and equivalent to the vectors $d\rho$ of (VIII.). (See Art. 124, p. 211.)

Ex. 1. Interpret Hamilton's focal and cyclic transformations of a self-conjugate function,

$$\phi\rho = a\alpha V\alpha\rho + b\beta S\beta\rho = g\rho + \lambda S\mu\rho + \mu S\lambda\rho,$$

where $\phi\rho$ represents the displacement due to a small pure strain.

[The terms may be taken separately. $a\alpha V\alpha\rho$ represents a shrinkage or an expansion to or from one line (α); $b\beta S\beta\rho$ represents an elongation parallel to another. See Minchin, *Treatise on Statics*, Art. 379.]

CHAPTER XIV.

DYNAMICS OF A PARTICLE.

ART. **114**. The rate of change of the momentum of a particle is equal to the applied force, or

$$\frac{d}{dt}\cdot m\dot{\rho}=m\ddot{\rho}=\xi \quad\text{.........................(I.)}$$

where m is the mass; $\dot{\rho}$ the velocity, $m\dot{\rho}$ the momentum and ξ the applied force.

The moment of momentum of the particle about any point A is

$$\mathrm{V}(\rho-\alpha)m\dot{\rho}=m\mathrm{V}(\rho-\alpha)\dot{\rho}; \quad\text{..................(II.)}$$

and if A is a fixed point the rate of change of moment of momentum is equal to the moment of the applied force, for

$$\frac{d}{dt}m\mathrm{V}(\rho-\alpha)\dot{\rho}=m\mathrm{V}(\rho-\alpha)\ddot{\rho}=\mathrm{V}(\rho-\alpha)\xi, \quad\text{.........(III.)}$$

since $\mathrm{V}\dot{\rho}\dot{\rho}=0$. If the point A is in motion with velocity $\dot{\alpha}$, the rate of change of moment of momentum is

$$m\mathrm{V}(\rho-\alpha)\ddot{\rho}-m\mathrm{V}\dot{\alpha}\dot{\rho}=\mathrm{V}(\rho-\alpha)\xi-m\mathrm{V}\dot{\alpha}\dot{\rho}, \quad\text{.........(IV.)}$$

and in this case it depends on the velocity of the point A and on that of the particle P, unless indeed the motion of A is constantly parallel to that of P.

Since $$\frac{d}{dt}\cdot\tfrac{1}{2}m\mathrm{T}\dot{\rho}^2=-m\mathrm{S}\dot{\rho}\ddot{\rho}=-\mathrm{S}\dot{\rho}\xi=-\frac{d}{dt}\int\mathrm{S}\xi d\rho, \quad\text{.........(V.)}$$

the energy equation is

$$\tfrac{1}{2}m\mathrm{T}\dot{\rho}^2+\int\mathrm{S}\xi d\rho=const.=E, \quad\text{..................(VI.)}$$

and for a conservative system of forces (Art. 56 (VII.), p. 74),

$$\int\mathrm{S}\xi d\rho=P,\quad \xi=-\nabla P. \quad\text{..................(VII.)}$$

Ex. 1. If the applied force is parallel to a fixed plane $S\lambda\rho=0$, deduce the integral $S\lambda\rho=at+b$; and if it is parallel to a fixed line (μ), show that $V\mu\rho=\alpha t+\beta$ where a, b, α and β are constants of integration.

Ex. 2. If the force is directed to a fixed centre—the origin of vectors ρ—show that

$$mV\rho\dot{\rho}=\beta=\text{a constant vector.}$$

Ex. 3. If T is the tangential and N the normal component of the force and v the velocity in any orbit, prove that if C is the curvature of the orbit,

$$v^2C=N,\quad \dot{v}=T.$$

[Letting accents denote deriveds of ρ with respect to the arc, we have $\dot{\rho}=\rho'v$, $\ddot{\rho}=\rho''v^2+\rho'\dot{v}$ since $v=\dot{s}$. Also $T\rho''=C$ and $\xi=\rho'T+U\rho''N$. See Art. 117.]

Art. **115.** The equation of motion of a particle of unit mass attracted to any number of fixed centres with forces varying as the distance is

$$\ddot{\rho}=\Sigma a_1(\alpha_1-\rho)=\Sigma a_1\alpha_1-\rho\Sigma a_1, \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(I.)}$$

the attraction to any centre being proportional to the distance $T(\alpha_1-\rho)$ and acting along $U(\alpha_1-\rho)$ towards the centre. The scalars a_1, a_2, etc., define the ratio of the magnitude of the attraction of the centres to the distance, and they are positive for attractive and negative for repulsive forces.

If α is the vector to the mean centre of the centres for the multiples $a_1, a_2, \ldots a_n$, and if a is the sum of the multiples, the equation takes the form

$$\ddot{\rho}=a(\alpha-\rho),\quad (a=\Sigma a_1,\ a\alpha=\Sigma a_1\alpha_1); \quad\ldots\ldots\ldots\ldots\ldots\text{(II.)}$$

and the particle moves as if attracted to the mean centre.

The more general equation

$$\ddot{\rho}+2b\dot{\rho}+c\rho=0, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(III.)}$$

where b and c are scalar constants, is that of the motion of a particle acted on by a force $(-c\rho)$ due to a centre at the origin attracting or repelling ($c>0$ or <0) proportionally to the distance, and also acted on by a force $(-2b\dot{\rho})$ proportional to the velocity and accelerating or retarding according as $b<0$ or >0.

To integrate this equation, we assume

$$\rho=\gamma_1e^{n_1t}+\gamma_2e^{n_2t}+\text{etc.} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(IV.)}$$

where γ_1, γ_2, etc., are constant vectors and n_1, n_2, etc., constant scalars, and we express that the result of substituting for ρ in (III.) is identically satisfied for all values of t. Equating to zero the coefficients of e^{n_1t}, etc., after substitution, we find

$$\gamma(n^2+2bn+c)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(V.)}$$

where γ and n stand for any one of the vectors γ_1 and the corresponding scalar n_1. These conditions require all but two of the vectors to vanish. The remaining two γ_1 and γ_2 are indeterminate, and the corresponding values of n are the roots of the coefficient of γ in (V.) and are

$$n_1 = -b+p,\ n_2 = -b-p,\ \text{if } p^2 = b^2 - c;$$
$$n_1 = -b+\sqrt{-1}q,\ n_2 = -b-\sqrt{-1}q,\ \text{if } q^2 = c-b^2; \ \ldots\text{(VI.)}$$

and the corresponding solution of the equation is

$$\rho = e^{-bt}(\gamma_1 e^{pt} + \gamma_2 e^{-pt}) \text{ or } \rho = e^{-bt}(\delta_1 \cos qt + \delta_2 \sin qt), \ \ldots\text{(VII.)}$$

the vectors γ_1 and γ_2 (or δ_1 and δ_2) being arbitrary constants of integration.

In the more general case, to solve the equation

$$\ddot{\rho} + \phi_1\dot{\rho} + \phi_2\rho = 0, \ \ldots\ldots\ldots\text{(VIII.)}$$

where ϕ_1 and ϕ_2 are two constant linear vector functions, and which represents a damped motion of a particle such as might be supposed to take place in a crystalline medium, an assumption of the form (IV.) gives

$$n^2\gamma + n\phi_1\gamma + \phi_2\gamma = 0, \ \ldots\ldots\ldots\text{(IX.)}$$

so that the function $\phi_2 + n\phi_1 + n^2$ has a zero root and γ is the corresponding axis. The third invariant of the function must vanish if it has a zero root, and the appropriate values of the scalars n are the roots of the equation

$$S(\phi_2 + n\phi_1 + n^2)\lambda(\phi_2 + n\phi_1 + n^2)\mu(\phi_2 + n\phi_1 + n^2)\nu = 0, \ \ldots\ldots\text{(X.)}$$

where λ, μ and ν are any vectors. Solving this equation we determine six linear functions with zero latent roots, and the corresponding axes (γ_1, γ_2, etc.), being determined, the solution is

$$\rho = \Sigma_1^6 \gamma_1 e^{n_1 t}, \ \ldots\ldots\ldots\text{(XI.)}$$

the arbitrary constants being the tensors of the vectors γ.

Ex. 1. Show how to determine the constants of integration.

[We may have given the initial position and the initial velocity—six constants. For example the solution of (II.) is $\rho = \alpha + \gamma_1 \cos\sqrt{a}t + \gamma_2 \sin\sqrt{a}t$, and if $\rho = \beta$ and $\dot{\rho} = \gamma$ when $t=0$, we have $\gamma_1 = \beta - \alpha$, $\gamma_2\sqrt{a} = \gamma$.]

ART. **116**. For a force directed to a fixed centre, the origin of vectors ρ,

$$\ddot{\rho} = \xi,\ U\xi = \pm U\rho, \ \ldots\ldots\ldots\text{(I.)}$$

and (Art. 114 (III.)) we deduce at once the integral of moment of momentum

$$V\rho\dot{\rho} = \beta, \ \ldots\ldots\ldots\text{(II.)}$$

where the constant β is double the vector area swept out by the radius vector in unit time. Conversely if the vector moment of momentum with respect to any fixed point is constant, that point is a centre to which the force acting on the particle is directed, for $\dot{\beta} = 0 = V\rho\ddot{\rho}$ or $\ddot{\rho} \parallel \xi \parallel \rho$. The orbit of the particle lies in the plane

$$S\rho\beta = 0. \ \ldots\ldots\ldots\text{(III.)}$$

In general the vector β admits of transformations such as the following (compare Art. 85 (II.), p. 132):

$$\beta = \mathrm{T}\rho^2 . \mathrm{V}\frac{\dot{\rho}}{\rho} = \mathrm{T}\rho^2 . \frac{d\mathrm{U}\rho}{dt} . \frac{1}{\mathrm{U}\rho} = \mathrm{T}\rho^2 . \omega = \mathrm{T}\rho^2 . \mathrm{U}\beta . \dot{w}, \ \text{...(IV.)}$$

where ω is the angular velocity of the radius vector, and where w (for a plane orbit) is the angle the radius vector makes with some prime vector or more generally where $\dot{w}$ is the scalar angular velocity. We may also write

$$\frac{\beta\mathrm{U}\rho}{\mathrm{T}\rho^2} = \frac{d\mathrm{U}\rho}{dt}, \text{ or } \frac{\mathrm{U}\rho}{\mathrm{T}\rho^2} = \beta^{-1} . \frac{d\mathrm{U}\rho}{dt}; \ \text{..............(V.)}$$

so that for a central force

$$\ddot{\rho} = -m\beta^{-1}\frac{d\mathrm{U}\rho}{dt}, \text{ if } \xi = -m\mathrm{U}\rho\mathrm{T}\rho^{-2}. \ \text{..............(VI.)}$$

In particular when the law of force is that of the inverse square, the scalar m is constant, and (VI.) integrates at once and gives

$$\dot{\rho} = -m\beta^{-1}\mathrm{U}\rho + \gamma \quad \text{where} \quad \mathrm{S}\beta\gamma = 0 \text{ by (II.)}, \ \text{.......(VII.)}$$

γ being a vector constant of integration. This shows that the *hodograph* of the motion is a circle whose centre is the extremity of the vector γ and whose radius is $m\mathrm{T}\beta^{-1}$.

Moreover, substituting for $\dot{\rho}$ in (II.), we find the equation of the orbit,

$$\beta = -m\beta^{-1}\mathrm{T}\rho + \mathrm{V}\rho\gamma; \ \text{..................(VIII.)}$$

which is equivalent to the two equations

$$m\mathrm{T}\rho = \mathrm{T}\beta^2 - \mathrm{S}\beta\gamma\rho, \quad \mathrm{S}\beta\rho = 0; \ \text{................(IX.)}$$

and which represents a conic referred to a focus as origin. If w is the angle the radius makes with the vector $\gamma\beta$ we may replace (IX.) by

$$\mathrm{T}\rho(1 + e\cos w) = p \quad \text{where} \quad e = m^{-1}\mathrm{T}\gamma\beta, \quad p = m^{-1}\mathrm{T}\beta^2, \ \text{...(X.)}$$

and e is the excentricity and p the semi-*latus-rectum*.

Taking the tensor of (VII.), utilizing (IX.) and observing that by (X.) $\mathrm{T}\gamma^2 = me^2p^{-1}$, we obtain the energy equation

$$\mathrm{T}\dot{\rho}^2 = \frac{2m}{\mathrm{T}\rho} - \frac{m}{a} \ \text{........................(XI.)}$$

where $a = p(1-e^2)^{-1}$ is the mean distance.

Now when we resolve the velocity along and perpendicular to ρ,

$$\dot{\rho} = \rho^{-1}\mathrm{S}\rho\dot{\rho} + \rho^{-1}\mathrm{V}\rho\dot{\rho} = \mathrm{U}\rho\dot{r} + \rho^{-1}\beta \quad \text{if} \quad r = \mathrm{T}\rho; \ \text{.....(XII.)}$$

whence on substitution in (XI.) we find

$$\dot{r}^2 = m\left(\frac{2}{r} - \frac{p}{r^2} - \frac{1}{a}\right), \quad \text{.....................(XIII.)}$$

which gives on integration the radius vector in terms of the time.

Ex. 1. Deduce the usual u and θ equations for a central orbit by expressing ρ in the form $rk^{\frac{2\theta}{\pi}}i$.

[Here $\dot{\rho} = (\dot{r} + r\dot{\theta}k)k^{\frac{2\theta}{\pi}}i$, $\beta = r^2\dot{\theta}k = hk$, $\dot{\theta} = hu^2$, $\dot{r} = r'\dot{\theta} = -hu'$ where accents denote differentiation with respect to θ and where $u = r^{-1}$. Thus

$$\dot{\rho} = -h(u' - uk)k^{\frac{2\theta}{\pi}}i, \quad \ddot{\rho} = -h^2u^2(u'' + u)k^{\frac{2\theta}{\pi}}i.]$$

Ex. 2. If α, β and γ are three unit vectors, α along the radius vector, γ perpendicular to the plane of the instantaneous orbit and $\beta = \gamma\alpha$; if $\dot{c}$ is the rate of description of angles by the radius vector in the orbit and if $\dot{a}$ is the rate at which the plane of the orbit turns round the radius vector, prove that the equation of motion is

$$\alpha(\ddot{r} - r\dot{c}^2) + \beta(2\dot{r}\dot{c} + r\ddot{c}) + \gamma r\dot{a}\dot{c} = \xi.$$

[Here $\frac{\dot{\alpha}}{\alpha} = \mathrm{V}\frac{\dot{\rho}}{\rho} = \gamma\dot{c}$, $\frac{\dot{\gamma}}{\gamma} = \mathrm{V}.\frac{\mathrm{V}\rho\ddot{\rho}}{\mathrm{V}\rho\dot{\rho}} = \alpha\dot{a}$, so that $\dot{\alpha} = \beta\dot{c}$, $\dot{\gamma} = -\beta\dot{a}$ and $\dot{\beta} = \gamma\dot{a} - \alpha\dot{c}$. Compare Art. 86. By the instantaneous orbit is meant the orbit which a planet would describe round the sun if the disturbing forces were suddenly removed. The equation exhibits the effect of the components of the force along and perpendicular to the radius vector and perpendicular to the plane of the orbit.]

Ex. 3. Express the equation of motion in a perturbed orbit in terms of the reciprocal of the radius vector (u), the rate of description of areas (h) and the rate (a') at which the orbit turns round the radius vector per unit description of angle in the orbit; and show that it is

$$(u'' + u)\alpha + \frac{h'u'}{h}\alpha - \frac{h'u}{h}\beta - ua'\gamma = -\frac{\xi}{h^2u^2}.$$

[We have to express everything in terms of $h = \mathrm{TV}\rho\dot{\rho} = r^2\dot{c}$, of u and of α and their differentials with respect to the angle c. Writing thus

$$\rho = \alpha u^{-1} \text{ we have } \dot{\rho} = hu^2.\frac{\partial}{\partial c}(\alpha u^{-1}) = hu^2(\beta u^{-1} - \alpha u'u^{-2}), \text{ etc.]}$$

Ex. 4. Express the equation of motion of a particle in the form

$$\alpha(u'' + u) + \alpha u'\frac{H'}{H} - \beta u\frac{H'}{H} - \gamma(us'' - su'') - \gamma\frac{H'}{H}(us' - su') = -\frac{\xi}{H^2u^2}$$

where u is the reciprocal of the projection of the radius vector on a fixed plane, α is a unit vector along this projection, γ is the unit normal to the plane, $\beta = \gamma\alpha$, H is the rate of description of the projection of areas, s is the tangent of the angle between the radius vector and the projection, and the independent variable is the angle in the fixed plane.

[Here $\rho = (\alpha + s\gamma)u^{-1}$, $\alpha'\alpha^{-1} = \gamma$, $\gamma' = 0$, $\beta' = -\alpha$, $Hu^2 = \dot{c}$ if c is the angle in the plane. The scalar equations to which the above is equivalent have been much used in the lunar theory.]

Ex. 5. Prove that the vector curvatures of an orbit and its hodograph are

$$\frac{d U d\rho}{d\rho} = V\frac{\xi}{\dot\rho}\frac{1}{T\dot\rho}, \quad \frac{d U d\dot\rho}{d\dot\rho} = V\frac{\dot\xi}{\xi}\cdot\frac{1}{T\ddot\rho},$$

and that for a central orbit they reduce to

$$\frac{d U d\rho}{d\rho} = \frac{\beta T\xi}{T\rho T\dot\rho^3}, \quad \frac{d U d\dot\rho}{d\dot\rho} = \frac{\beta}{T\xi T\rho^2}$$

where $\beta = V\rho\dot\rho$.

(*a*) Hence the law of nature is the only law for which the hodograph is a circle for all initial conditions.

ART. **117**. The equation of motion of a particle constrained to move along a curve or on a surface is

$$\ddot\rho = \xi + \nu \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I.)}$$

where ν is the reaction arising from the constraint. If there is no friction, the reaction is at right angles to the direction of motion or the vector ν lies in the normal plane of the constraining curve or is the normal to the constraining surface. The condition

$$S\nu\dot\rho = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II.)}$$

which is then satisfied, allows us to retain the equations (V.) and (VI.) of Art. 114.

In terms of the deriveds with respect to the arc s of the orbit which we now denote by ρ', ρ'', etc., we have (compare Art. 85, Ex. 1, p. 133),

$$\dot\rho = \rho' v, \quad \ddot\rho = \rho'' v^2 + \rho'\dot v, \quad v = \dot s, \quad \dot v = v'v, \quad \ldots\ldots\ldots\text{(III.)}$$

or in the notation of Art. 86, p. 134,

$$\dot\rho = \alpha v, \quad \ddot\rho = \beta c_1 v^2 + \alpha\dot v \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(IV.)}$$

where v is the velocity; and the equation of motion is

$$\rho'' v^2 + \rho'\dot v = \xi + \nu. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(V.)}$$

In the case of a constraining curve, the motion must be determined from the energy equation which is alone available for this purpose. For a surface we have, on elimination of the unknown tensor of ν,

$$V(\ddot\rho - \xi)\nu = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(VI.)}$$

and in this equation ν is proportional to a known function of ρ —the result of operating by ∇ on the scalar equation of the constraining surface. (Art. 54, p. 69.)

If on the other hand we seek the reaction arising from the curve or surface, we have by (II.)

$$\nu = \rho'^{-1}V\rho'\nu = \rho'' v^2 - \rho'^{-1}V\rho'\xi = -2\rho''\int S\xi d\rho - \rho'^{-1}V\rho'\xi, \quad \ldots\text{(VII.)}$$

the energy equation being employed in the last transformation.

For a rough constraint, the equation of motion may be written in the form

$$\ddot{\rho}=\rho''v^2+\rho'vv'=\xi+\nu-n\rho'\mathrm{T}\nu,\quad \mathrm{S}\rho'\nu=0, \quad\text{.................(VIII.)}$$

where n is the coefficient of friction.

Resolving along and at right angles to ρ' this equation gives

$$vv'+\mathrm{S}\rho'\xi=-n\mathrm{T}\nu,\quad \nu=\rho''v^2-\rho'^{-1}\mathrm{V}\rho'\xi\ ; \quad\text{.................(IX.)}$$

whence on elimination of $\mathrm{T}\nu$,

$$\rho''v^2-\rho'^{-1}\mathrm{V}\rho'\xi=-\mathrm{U}\nu\,.\,n^{-1}(vv'+\mathrm{S}\rho'\xi)\ ; \quad\text{.................(X.)}$$

or again in terms of the vectors $\dot{\rho}$ and $\ddot{\rho}$, we have

$$n\mathrm{V}\dot{\rho}(\ddot{\rho}-\xi)=\mathrm{U}(\dot{\rho}\nu)\,.\,\mathrm{S}\dot{\rho}(\ddot{\rho}-\xi), \quad\text{.................(XI.)}$$

because $\mathrm{S}\dot{\rho}(\ddot{\rho}-\xi)=n\mathrm{T}(\dot{\rho}\nu)$ and $\mathrm{UV}\dot{\rho}(\ddot{\rho}-\xi)=\mathrm{U}(\dot{\rho}\nu)$. Equation (X.) or (XI.) may be employed for a constraining surface. In the case of a curve we must take the tensor of each side to eliminate the unknown $\mathrm{U}\nu$. We may remark that it follows from (IX.) that if the curve is a geodesic on the constraining surface $\mathrm{V}\,.\,\nu\rho'^{-1}\mathrm{V}\rho'\xi=0$ or

$$\mathrm{S}\nu\rho'\xi=0, \quad\text{.................(XII.)}$$

because (Art. 90) for a geodesic $\rho''\parallel\nu$. In other words, when the direction of the applied force is coplanar with the normal to the surface and the tangent to the orbit, the curve is a geodesic on the surface, and in particular this is the case when there is no applied force.

If the constraining curve or surface is in motion so that, Art. 104, p. 168, the vector ρ to the particle from a fixed point is connected with the vector ϖ to the particle from a point moving with the constraint by the equation

$$\rho=\tau+q\varpi q^{-1}, \quad\text{.................(XIII.)}$$

in which τ and q are supposed to be given functions of t, the equation of motion takes the form (compare Art. 105, p. 171)

$$\ddot{\tau}+q(\ddot{\varpi}+2\mathrm{V}\iota\dot{\varpi}+\mathrm{V}\dot{\iota}\varpi+\mathrm{V}\iota\mathrm{V}\iota\varpi)q^{-1}=\xi+\nu, \quad\text{.................(XIV.)}$$

and for a smooth constraint,

$$\mathrm{S}q\dot{\varpi}q^{-1}\nu=0, \quad\text{.................(XV.)}$$

$q\dot{\varpi}q^{-1}$ being the velocity with which the particle moves along the curve or surface of the constraint.

Ex. 1. A particle moves under gravity on a surface of revolution having its axis vertical.

[If k is the unit vector directed vertically downwards, the equation of motion is $\mathrm{V}(\ddot{\rho}-gk)\nu=0$. Since the surface is of revolution, the vectors ν, k and ρ are coplanar, or $\mathrm{S}\rho k\nu=0$, so that $\mathrm{V}k\rho\parallel\mathrm{V}k\nu\parallel\mathrm{V}\nu\rho$. Operating on the equation of motion by $\mathrm{S}k$ or $\mathrm{S}\rho$ we find the integrable relation $\mathrm{S}k\rho\ddot{\rho}=0$, so that $\mathrm{S}k\rho\dot{\rho}=-h$ where h is the constant rate of description of area by the projection of ρ on the horizontal plane. We have also $\mathrm{S}\nu\dot{\rho}=0$ and $\mathrm{S}k\dot{\rho}=-\dot{z}$ if we write $\mathrm{S}k\rho=-z$. From these three equations $\dot{\rho}\mathrm{SV}k\rho\mathrm{V}k\nu=-h\mathrm{V}k\nu-\dot{z}\mathrm{V}\nu\mathrm{V}k\rho$; and if the equation of the surface is given in the form $\mathrm{T}\rho=f(z)=f(-\mathrm{S}k\rho)$ we may put $\nu=\mathrm{U}\rho-kf'(z)$ and $\mathrm{V}k\rho=\mathrm{V}k\nu f(z)$. Hence $\dot{\rho}^2\mathrm{V}k\rho^2=h^2-\dot{z}^2\nu^2\mathrm{T}\rho^2$; and by Art. 114 (VI.) on expressing everything in terms of z we obtain the equation

$$\dot{z}^2(f^2-2zff'+f^2f'^2)=2(E+gz)(f^2-z^2)-h^2.$$

If the surface is spherical $f(z)$ is constant and equal to the radius of the sphere, so that f' is zero.

Again if w is the angle the plane of ρ and k makes with some initial plane,

$$h=\dot{w}\mathrm{TV}k\rho^2=\dot{w}(f^2-z^2)$$

or

$$\frac{\mathrm{d}w}{\mathrm{d}z}\cdot\dot{z}(f^2-z^2)=h,$$

from which w can be found in terms of z by the previous equation.

If, on the other hand, the equation of the surface is given in the form $\mathrm{S}k\rho=f(\mathrm{T}\rho)$, it may be more convenient to obtain an equation in $r(=\mathrm{T}\rho)$ and $\dot{r}$ by using $\mathrm{S}\rho\dot{\rho}+r\dot{r}=0$ instead of $\mathrm{S}k\dot{\rho}+\dot{z}=0$; and if the equation is of the form $\mathrm{S}k\rho=f(\mathrm{TV}k\rho)=f(p)$ we may use $\mathrm{SV}k\rho\mathrm{V}k\dot{\rho}+p\dot{p}=0$.]

Ex. 2. A particle slides under gravity within a fine smooth tube which revolves round a vertical axis.

[The origin being taken on the axis, the vector to the particle is $\rho=q\varpi q^{-1}$ (compare p. 168), and if n is the angle through which the tube has been rotated from some initial position,

$$\dot{\rho}=q(\dot{\varpi}+\dot{n}\mathrm{V}k\varpi)q^{-1},\quad \ddot{\rho}=q(\ddot{\varpi}+2\dot{n}\mathrm{V}k\dot{\varpi}+\dot{n}^2k\mathrm{V}k\varpi+\ddot{n}\mathrm{V}k\varpi)q^{-1};$$

while the equation of motion is $\ddot{\rho}=gk+\nu$ where $\mathrm{S}\nu q\dot{\varpi}q^{-1}=0$. Because the axis of q is parallel to k, we find on elimination of the reaction ν,

$$\mathrm{S}\dot{\varpi}(\ddot{\varpi}+\dot{n}^2k\mathrm{V}k\varpi+\ddot{n}\mathrm{V}k\varpi)=g\mathrm{S}\dot{\varpi}k;$$

and in this equation $\dot{n}$ and $\ddot{n}$ are given functions of t when the law of rotation is known, and ϖ is a known function of a parameter variable with the time when the form of the tube is known. If the velocity of rotation is uniform, the equation integrates and

$$\tfrac{1}{2}(\dot{\varpi}^2+\dot{n}^2\mathrm{V}k\varpi^2)=g\mathrm{S}k\varpi+\tfrac{1}{2}C.$$

If for example the curve is a helix with its axis vertical so that $\varpi=a(i\cos u+j\sin u)+bku$ we have $\dot{\varpi}^2=-(a^2+b^2)\dot{u}^2$, and $\mathrm{V}k\varpi^2=-a^2$, and the equation is $\dot{u}^2(a^2+b^2)+\dot{n}^2a^2=2gbu-C$; and if the curve is a vertical circle, $\varpi=a(i\cos u+k\sin u)$ we have

$$\dot{u}^2a^2+\dot{n}^2a^2\cos^2 u=2ga\sin u-C.]$$

Ex. 3. A particle under gravity traverses with uniform velocity a smooth curve which rotates uniformly round a vertical axis. Prove that the curve lies on a paraboloid of revolution.

[The equation of the surface on which the curve must lie is

$$\dot{n}^2\mathrm{TV}k\varpi^2+2g\mathrm{S}k\varpi=const.]$$

Ex. 4. Two particles of masses m and m', connected by an inextensible string which remains stretched throughout the motion, are projected from the extremities of the vectors α and α' with the velocities β and β'; prove that the vector to the particle m during the motion is ρ where

$$\begin{aligned}\rho(m+m')&=m(\alpha+\beta t)+m'(\alpha'+\beta' t)\\&\quad+m'\mathrm{T}(\alpha-\alpha').(\mathrm{U}(\alpha-\alpha')\cos nt+\mathrm{U}(\beta-\beta')\sin nt),\end{aligned}$$

the scalar n being defined by

$$n\mathrm{T}(\alpha-\alpha')=\mathrm{T}(\beta-\beta').$$

Ex. 5. If a particle can be made by suitable initial conditions to describe a given curve under the action of a force ξ, show that

$$2\rho''\int\mathrm{S}\xi\mathrm{d}\rho-\rho'\mathrm{V}\rho'\xi=0,$$

ρ' and ρ'' being the first and second deriveds with respect to the arc and a suitable constant being included in the integral which is taken along the curve.

(*a*) Hence deduce M. Bonnet's theorem.

[We have $m(\rho''v^2+\rho'vv')=\xi$ which gives $mvv'=-S\xi\rho'$ and $mv^2=-2\int S\xi d\rho$, etc. Conversely if the condition is satisfied it follows that a particle will for suitable initial conditions describe the curve. If ξ_1, ξ_2, etc., are forces under which, acting separately, a particle can describe the curve, and if for greater clearness we replace $\int S\xi_n d\rho$ by $C_n+\int_0 S\xi_n d\rho$ (the new integral being taken from any selected point on the curve), we have

$$\Sigma\{2\rho''(C_n+\int_0 S\xi_n d\rho)-\Sigma\rho' V\rho'\xi_n\}=2\rho''(\Sigma C_n+\int_0 S\,.\,\Sigma\xi_n\,.\,d\rho-\rho' V\rho'\Sigma\xi_n);$$

or a particle will describe the curve freely under the action of the resultant of the forces provided its mass m and the velocity v satisfy $mv^2=\Sigma m_n v_n^2$ initially.]

Ex. 6. Show that the condition of the last example is equivalent to the conditions

$$S\rho'\rho''\xi=0,\quad \frac{d}{ds}S\rho''^{-1}\xi+2S\rho'\xi=0,$$

which assert that the force must be in the osculating plane of the curve, and that the rate of change (as we pass along the curve) of the product of the radius of curvature into the normal component of the force is equal to double the tangential component.

ART. **118**. Tait has applied the calculus of variations in the following manner in the determination of the curves of quickest descent, or the brachistochrones, for a conservative system of forces. (*Quaternions*, Arts. 518 and 523.)

If the integral $$A=\int Q\,.\,Td\rho=\int Q\,.\,ds \quad \text{(I.)}$$

is taken along a curve, Q being a given scalar function of ρ, the variation of the integral corresponding to a variation of the curve is

$$\delta A=\int\delta Q\,.\,Td\rho+\int Q\,.\,\delta Td\rho=-\int S\delta\rho\nabla\,.\,Q\,.\,Td\rho-\int QSUd\rho\,.\,\delta d\rho.$$

The symbols d and δ are commutative in order of operation, so that on integrating by parts

$$\int QSUd\rho\,.\,\delta d\rho=\int QSUd\rho\,.\,d\delta\rho=[QSUd\rho\,.\,\delta\rho]-\int S\delta\rho d(QUd\rho)$$

where the term in square brackets corresponds to the variation of the limits of the integral. Thus

$$\delta A=-[QSUd\rho\,.\,\delta\rho]+\int S\delta\rho\{d(QUd\rho)-\nabla Q\,.\,Td\rho\}\quad \text{(II.)}$$

If the integral is stationary, the variation vanishes and the term under the sign of integration in (II.) must be zero for all vectors $\delta\rho$. And since $\delta\rho$ may have any direction when the curve is not restricted in any manner except at the limits, we must have

$$\mathrm{d}(Q\mathrm{Ud}\rho) - \nabla Q\,.\,\mathrm{Td}\rho = 0, \text{ or } \frac{\mathrm{d}}{\mathrm{d}s}(Q\rho') - \nabla Q = 0. \quad \text{......(III.)}$$

If on the other hand the curve is constrained to lie on a surface so that $\mathrm{S}\nu\delta\rho = 0$ where ν is normal to the surface, the condition is

$$\mathrm{V}\nu\left(\frac{\mathrm{d}}{\mathrm{d}s}(Q\rho') - \nabla Q\right) = 0. \quad \text{......................(IV.)}$$

For the brachistochrone the integral A is the time of description of the curve or

$$A = t = \int v^{-1}\,.\,\mathrm{d}s, \quad Q = \mathrm{T}\dot\rho^{-1} = (2E - 2P)^{-\frac{1}{2}} \quad \text{..........(V.)}$$

by Art. 114 (VI.), so that $\nabla Q = \nabla P\,.\,Q^3 = \nabla P\,.\,\mathrm{T}\dot\rho^{-3}$. The first equation (III.) now becomes

$$\mathrm{d}(\mathrm{T}\dot\rho^{-1}\,.\,\mathrm{Ud}\rho) - \nabla P\,.\,\mathrm{T}\dot\rho^{-2}\,.\,\mathrm{d}t = 0 \quad \text{or} \quad \mathrm{d}\,.\,\dot\rho^{-1} + \nabla P\,.\,\mathrm{T}\dot\rho^{-2}\mathrm{d}t = 0,$$

or finally
$$\ddot\rho + \dot\rho^{-1}\,.\,\nabla P\,.\,\dot\rho = 0. \quad \text{........................(VI.)}$$

Tait remarks "It is very instructive to compare this equation with that of the free path ($\ddot\rho + \nabla P = 0$); noting how the force $-\nabla P$ is, as it were, *reflected* on the tangent of the path."

Ex. Determine the brachistochrone when gravity is the only force.

[Here $\nabla P = -\kappa$, a constant vector, and the equation $\mathrm{d}\dot\rho^{-1} - \kappa\mathrm{T}\dot\rho^{-2}\mathrm{d}t = 0$ shows that $\dot\rho^{-1} = \alpha + \kappa f(t)$ where α is a constant vector which may without loss of generality be supposed to be perpendicular to κ. Substitution gives $\mathrm{d}f - (\mathrm{T}\alpha^2 + \mathrm{T}\kappa^2 f^2)\mathrm{d}t = 0$, and the solution of this is

$$f = \mathrm{T}\,.\,\kappa^{-1}\alpha \tan \mathrm{T}\alpha\kappa(t - t_0) = \mathrm{T}\,.\,\kappa^{-1}\alpha \tan n(t - t_0)$$

where $n = \mathrm{T}\,.\,\alpha\kappa$. Thus

$$\dot\rho^{-1} = -\mathrm{T}\alpha(\mathrm{U}\alpha + \mathrm{U}\kappa \tan n(t - t_0))$$

and
$$\dot\rho = \mathrm{T}\alpha^{-1}\,.\,\cos^2 n(t - t_0)(\mathrm{U}\alpha + \mathrm{U}\kappa \tan n(t - t_0)),$$

and on integration

$$\rho = \beta - \tfrac{1}{4}n^{-1}\mathrm{T}\alpha^{-1}[\mathrm{U}\alpha\{2n(t - t_0) + \sin 2n(t - t_0)\} - \mathrm{U}\kappa \cos 2n(t - t_0)]$$

which represents a cycloid. (Tait's *Quaternions*, Art. 524.)]

CHAPTER XV.

DYNAMICS.

ART. **119**. Let m_1, m_2, etc., be the masses of particles of any dynamical system which are situated at the extremities of the vectors ρ_1, ρ_2, etc., drawn from a fixed origin. By Newton's second law the equation of motion of the particle m_1 is

$$m_1\ddot{\rho}_1 = \xi_1 + \xi_{12} + \xi_{13} + \text{etc.}, \qquad \text{(I.)}$$

where ξ_1 is the force external to the system which acts on m_1 and where ξ_{12} is the force due to the interaction of m_2 on m_1, etc. By Newton's third law action and reaction are equal and opposite, or

$$\xi_{12} + \xi_{21} = 0, \quad \mathrm{V}\rho_1\xi_{12} + \mathrm{V}\rho_2\xi_{21} = 0, \qquad \text{(II.)}$$

these being the conditions that ξ_{12} and ξ_{21} should equilibrate. Hence by adding equations such as (I.) for all the particles, and by adding the results of operating on these equations by $\mathrm{V}\rho_1$, $\mathrm{V}\rho_2$, etc., we obtain the equations

$$\Sigma m_1\ddot{\rho}_1 = \Sigma\xi_1, \quad \Sigma m_1\mathrm{V}\rho_1\ddot{\rho}_1 = \Sigma\mathrm{V}\rho_1\xi_1, \qquad \text{(III.)}$$

which are independent of the interactions of the particles.

Attending to (II.) the rate of change of kinetic energy of the system of particles is evidently

$$\frac{d}{dt}\cdot\tfrac{1}{2}\Sigma m_1\mathrm{T}\dot{\rho}_1{}^2 = -\Sigma m_1\mathrm{S}\dot{\rho}_1\ddot{\rho}_1 = -\Sigma\mathrm{S}\dot{\rho}_1\xi_1 - \Sigma\mathrm{S}(\dot{\rho}_1 - \dot{\rho}_2)\xi_{12}, \ldots \text{(IV.)}$$

and because (II.) implies $\xi_{12} \parallel \rho_1 - \rho_2$ we see that this is independent of the interactions provided the relative velocity of every pair of particles is at right angles to the line joining them—or in other words, provided the distance between every pair of particles remains unchanged.

Writing

$$M = \Sigma m_1, \quad M\rho = \Sigma m_1\rho_1, \quad \xi = \Sigma\xi_1, \quad \eta = \Sigma\mathrm{V}\rho_1\xi_1, \quad \theta = \Sigma m_1\mathrm{V}\rho_1\dot{\rho}_1, \text{(V.)}$$

so that M is the total mass of the system, ρ the vector to the centre of mass, ξ the resultant external force, η the resultant moment of the external forces with respect to the origin as base point and θ the resultant moment of momentum with respect to the origin, the equations (III.) become

$$M\ddot{\rho}=\xi, \quad \dot{\theta}=\eta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(VI.)}$$

When the external forces are zero, ξ and η vanish and the integrals of (VI.) are

$$M\rho=\alpha t+\beta, \quad \theta=\gamma, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(VII.)}$$

where α, β and γ are constant vectors; and when the internal forces are given functions of the distances between the particles, we have also in this case the integral of energy

$$\tfrac{1}{2}\Sigma m_1 T\dot{\rho}_1{}^2=\Sigma f\,.\,T(\rho_1-\rho_2) \text{ where } \xi_{12}=U(\rho_1-\rho_2)f'\,.\,T(\rho_1-\rho_2). \text{(VIII.)}$$

ART. **120**. With reference to a point moving in any arbitrary manner, the extremity of the vector ϵ, the moment of momentum is

$$\theta_\epsilon=\Sigma m_1 V(\rho_1-\epsilon)(\dot{\rho}_1-\dot{\epsilon})=\theta-MV(\rho\dot{\epsilon}+\epsilon\dot{\rho}-\epsilon\dot{\epsilon})\,; \quad \ldots\ldots\text{(I.)}$$

and (VI.), Art. 119, may be replaced by

$$M\ddot{\rho}=\xi, \quad \dot{\theta}_\epsilon=\eta_\epsilon-MV(\rho-\epsilon)\ddot{\epsilon}, \ldots\ldots\ldots\ldots\ldots\ldots\text{(II.)}$$

where $\eta_\epsilon=\eta-V\epsilon\xi$ is the resultant moment of the forces about the extremity of ϵ. In particular when ϵ terminates at the centre of mass, the equations are

$$M\ddot{\rho}=\xi, \quad \dot{\theta}_0=\eta_0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(III.)}$$

where θ_0 and η_0 refer to the centre of mass. These equations are of the same form as those of the last article. We may note that in general

$$\theta_0=\theta-MV\rho\dot{\rho}=\theta_\epsilon-MV\rho_\epsilon\dot{\rho}_\epsilon, \ldots\ldots\ldots\ldots\ldots\ldots\text{(IV.)}$$

where $\rho_\epsilon=\rho-\epsilon$.

Ex. 1. Find the locus of points fixed in space about which at any instant the moment of momentum is a minimum.

[If the extremity of ϵ terminates at a fixed point $\theta_\epsilon=\theta-MV\epsilon\dot{\rho}$, and the locus of points for which $T\theta_\epsilon$ has a given value is the right circular cylinder $T(\theta-MV\epsilon\dot{\rho})=T\theta_\epsilon$. Writing $\theta=M(p\dot{\rho}+V\epsilon_0\dot{\rho})$ we have

$$T\theta_\epsilon{}^2=M^2p^2T\dot{\rho}^2+M^2TV(\epsilon-\epsilon_0)\dot{\rho}^2.$$

The locus is the line $MV\epsilon\dot{\rho}=V\theta\dot{\rho}\,.\,\dot{\rho}^{-1}$. Compare Art. 99, p. 156.]

Ex. 2. A point moves in such a manner that the moment of momentum with respect to it is constant. Determine the particulars of the motion.

[If θ_ϵ is constant, the relation (IV.) $MV\rho_\epsilon\dot{\rho}_\epsilon=-\theta+\theta_\epsilon+MV\rho\dot{\rho}$ gives, on differentiating twice and utilizing the equations of motion (Art. 119 (VI.)),

$$MV\rho_\epsilon\ddot{\rho}_\epsilon=-\eta+V\rho\xi, \quad MV(\rho_\epsilon\dddot{\rho}_\epsilon+\dot{\rho}_\epsilon\ddot{\rho}_\epsilon)=-\dot{\eta}+V\dot{\rho}\xi+V\rho\dot{\xi}$$

because θ_ϵ is constant. Forming the vectors of the products of right and of left hand members of the first and second of these three relations, and also forming the scalar of the product of corresponding members of the three relations, we obtain the equation

$$\rho_\epsilon=\rho-\epsilon=\pm V(\theta-\theta_\epsilon-MV\rho\dot{\rho})(\eta-V\rho\xi)$$
$$\times\{MS(\theta-\theta_\epsilon-MV\rho\dot{\rho})(\eta-V\rho\xi)(\dot{\eta}-V\rho\dot{\xi}-V\dot{\rho}\xi)\}^{-\frac{1}{2}},$$

so that ϵ is expressed in terms of quantities which are known when the motion of the system is given. There are thus two paths corresponding to the double sign symmetrically placed with respect to the path of the centre of mass.]

Ex. 3. Refer the equations of motion to variable axes.
[See Art. 105 and the formulae of differentiation (VI.) and (XI.), p. 170.]

ART. **121**. In the case of a rigid body, let ϵ be the vector to any point fixed in it and let ω be the angular velocity. Then by Art. 105, p. 170,

$$\dot{\rho}_1 - \dot{\epsilon} = V\omega(\rho_1 - \epsilon), \quad \text{.........................(I.)}$$

because the velocity of the point in the body at the extremity of ρ_1 relatively to that at the extremity of ϵ is due to the angular velocity ω. Equation (I.) of the last article may now be replaced by

$$\theta_\epsilon = \Sigma m_1 V(\rho_1 - \epsilon) V\omega(\rho_1 - \epsilon) = \phi_\epsilon \omega, \quad \text{...............(II.)}$$

so that θ_ϵ is a linear function of ω. The linear function ϕ_ϵ is fixed relatively to the body because the vectors $\rho_1 - \epsilon$, etc., are fixed in the body, but in considering the rate of change of $\phi_\epsilon \omega$ we must take account of the change of orientation of the body as well as of the change of ω. We have (Art. 105 (IX.)),

$$\frac{d}{dt}\phi_\epsilon\omega = \frac{\partial(\phi_\epsilon\omega)}{\partial t} + V\omega\phi_\epsilon\omega = \phi_\epsilon\dot{\omega} + V\omega\phi_\epsilon\omega; \quad \text{...........(III.)}$$

and equations (II.) of the last article become

$$M\ddot{\rho} = \xi, \quad \phi_\epsilon\dot{\omega} + V\omega\phi_\epsilon\omega = \eta_\epsilon - MV(\rho - \epsilon)\ddot{\epsilon}; \quad \text{........(IV.)}$$

and when ϵ terminates at the centre of mass (Art. 120 (III.)),

$$M\ddot{\rho} = \xi, \quad \phi\dot{\omega} + V\omega\phi\omega = \eta_0, \quad \text{.....................(V.)}$$

if (Art. 120 (IV.)) $\phi\omega = \phi_\epsilon\omega - MV(\rho - \epsilon)V\omega(\rho - \epsilon)$ refers to the centre of mass.

If the body has a fixed point, the extremity of ϵ, (IV.) reduces to

$$\phi_\epsilon\dot{\omega} + V\omega\phi_\epsilon\omega = \eta_\epsilon. \quad \text{........................(VI.)}$$

In general the vector $\phi_\epsilon\omega$ is the moment of momentum of the body with reference to the fixed point which instantaneously coincides with the extremity of the vector ϵ, and the moment of inertia round any line ($U\omega$) through that point is

$$S\omega^{-1}\phi_\epsilon\omega = \Sigma m_1 TV \,.\, U\omega \,.\, (\rho_1 - \epsilon)^2, \quad \text{...............(VII.)}$$

and this is numerically equal to the reciprocal of the square of the parallel radius of the quadric

$$S\varpi\phi_\epsilon\varpi = -1. \quad \text{........................(VIII.)}$$

The function ϕ_ϵ may be called the *inertia function* corresponding to the extremity of ϵ.

The principal axes of the body through the point are the axes of the self-conjugate function ϕ_ϵ, and the moment of inertia round a principal axis is maximum, or minimum, or at least stationary in value. If the extremity of ϵ is fixed in space as well as in the body, so that the body moves about a fixed point, it appears from (VI.) that when the body is set rotating under no forces about one of these principal axes, it will rotate permanently round it. For we have $V\omega\phi_\epsilon\omega=0$ if ω is along a principal axis, and $\phi_\epsilon\dot{\omega}=0$ by (VI.); hence $\dot{\omega}=0$ since the function has not in general a zero root.

The energy equation (Art. 119 (IV.)) easily reduces in terms of $\dot{\epsilon}$ and ω to

$$\frac{d}{dt}\{\tfrac{1}{2}MT\dot{\epsilon}^2-MS\dot{\epsilon}V\omega(\rho-\epsilon)-\tfrac{1}{2}S\omega\phi_\epsilon\omega\}=-S\dot{\epsilon}\xi-S\omega\eta_\epsilon,\ \ldots(\text{IX.})$$

where $\eta_\epsilon=\Sigma V(\rho_1-\epsilon)\xi_1$; and when ϵ terminates at the centre of mass

$$\frac{d}{dt}(\tfrac{1}{2}MT\dot{\rho}^2-\tfrac{1}{2}S\omega\phi\omega)=-S\dot{\rho}\xi-S\omega\eta_0.\ \ldots\ldots\ldots\ldots\ldots(\text{X.})$$

Ex. 1. Prove the relation (III.) by direct differentiation of the explicit form

$$\phi_\epsilon\omega=\Sigma m_1V(\rho_1-\epsilon)V\omega(\rho_1-\epsilon).$$

[We have $$\frac{d}{dt}.V.(\rho_1-\epsilon)V\omega(\rho_1-\epsilon)$$

$$=V.V\omega(\rho_1-\epsilon)V\omega(\rho_1-\epsilon)+V.(\rho_1-\epsilon)V\dot{\omega}(\rho_1-\epsilon)+V.(\rho_1-\epsilon)V\omega V\omega(\rho_1-\epsilon)$$

by (I.). The first term on the right vanishes. The third is

$$V\omega(\rho_1-\epsilon)S\omega(\rho_1-\epsilon)\ \text{or}\ V.\omega V(\rho_1-\epsilon)V\omega(\rho_1-\epsilon).]$$

Ex. 2. If I is a principal moment of inertia at the extremity of the vector ϵ, or in other words a latent root of ϕ_ϵ, show that

$$I^3-2n''I^2+(n''^2+n')I-(n''n'-n)=0,$$

where n'', n' and n are three positive scalars, namely,

$$n''=-\Sigma m_1(\rho_1-\epsilon)^2;\ n'=-\Sigma m_1m_2V(\rho_1-\epsilon)(\rho_2-\epsilon)^2;$$
$$n=\Sigma m_1m_2m_3S(\rho_1-\epsilon)(\rho_2-\epsilon)(\rho_3-\epsilon)^2.$$

[See *Elements*, Art. 417, and observe that $\phi_\epsilon\omega=n''\omega+\Sigma m_1(\rho_1-\epsilon)S\omega(\rho_1-\epsilon)$. Compare Art. 65, Ex. 1, p. 92.]

Ex. 3. The function ϕ_ϖ corresponding to the extremity of the vector ϖ drawn from the centre of mass is

$$\phi_\varpi\omega=\phi\omega+MV\varpi V\omega\varpi,$$

where ϕ corresponds to the centre of mass; the principal axes at the extremity of ϖ are the normals to the three confocals.

$$S\varpi(M^{-1}\phi-u)^{-1}\varpi=-1,$$

which pass through that point; and the locus of points at which one of the moments of inertia is equal to I is the quartic surface

$$S\varpi(M^{-1}\phi+T\varpi^2-M^{-1}I)^{-1}\varpi=-1.$$

[If $\phi_\varpi\alpha=I\alpha=\phi\alpha+MV\varpi V\alpha\varpi$, we have $(\phi+MT\varpi^2-I)\alpha+M\varpi S\alpha\varpi=0$, etc., and $u=M^{-1}I-T\varpi^2$. Compare Art. 101 (XX.) and (XXI.), p. 162.]

Ex. 4. A body under no applied forces moves about a fixed point. The equation of motion $\phi\dot{\omega}+V\omega\phi\omega=0$, furnishes the integrals

$$\phi\omega=\theta,\ S\omega\phi\omega=-h^2$$

where θ and h are constants of integration. Interpret this result.

(*a*) The equation $S\omega\phi\omega=-h^2$ may be regarded as representing an ellipsoid fixed in the body which rolls upon a plane fixed in space, and represented by the equation $S\omega\theta=-h^2$, the point of contact being the extremity of the vector ω.

(*b*) The equation $\theta^2 S\omega\phi\omega-h^2\phi\omega^2=0$ represents a cone fixed in the body which is the body locus of the instantaneous axis of rotation; and because the rate of change of ω is the same with respect to the body as with respect to lines of reference fixed in space (Art. 105 (x.)) it follows that this cone rolls on the space locus of the instantaneous axis.

(*c*) The extremity of the vector ω describes in the body part of the curve of intersection of two quadrics fixed in the body (the *polhode*) $S\omega\phi\omega=-h^2$ and $S\omega\phi^2\omega=\theta^2$, and the locus of the same point in space is a plane curve (the *herpolhode*).

(*d*) The vector θ, though fixed in space, describes in the body the cone $g^2 S\theta\phi^{-1}\theta-h^2\theta^2=0$ where $g=T\theta$ is the constant tensor of θ, and the extremity of θ traces out part of the sphero-conic in which this cone cuts the reciprocal quadric $S\theta\phi^{-1}\theta=-h^2$.

(*e*) The reciprocal quadric, fixed in the body, passes through a fixed point in space, and the central perpendicular on the tangent plane at this point varies inversely as the angular velocity.

(*f*) The relations

$$S\phi\omega(\ddot{\omega}+V\omega\dot{\omega})=0,\quad S\phi\dot{\omega}(\dot{\omega}-V\omega\dot{\omega})=0,$$

in which $\ddot{\omega}$ is the rate of change of $\dot{\omega}$ with respect to the body, may be obtained by differentiating the equation of motion. Hence

$$\phi\omega\,.\,(S\omega\dot{\omega}\ddot{\omega}+V\omega\dot{\omega}^2)=-h^2V\dot{\omega}(\ddot{\omega}+V\omega\dot{\omega})$$

and $$\phi\dot{\omega}(S\omega\dot{\omega}\ddot{\omega}+V\omega\omega^2)=h^2V\omega V\dot{\omega}(\ddot{\omega}+V\omega\dot{\omega});$$

and the vectors ω, $\dot{\omega}$ and $\ddot{\omega}$ satisfy a condition

$$SV\omega(\ddot{\omega}-V\omega\dot{\omega})V\dot{\omega}(\ddot{\omega}+V\omega\dot{\omega})=0,$$

which is independent of the constants of the body. The corresponding relation $$SV\omega(D_t^2\omega-2V\omega D_t\omega)VD_t\omega D_t^2\omega=0$$

connects ω and its first and second deriveds $D_t\omega$ and $D_t^2\omega$ with respect to fixed axes.

(*g*) Knowing ω at any instant and its first and second deriveds with reference either to axes fixed in the body or in space, the function ϕ is determinate to a factor.

Ex. 5. The angular velocity of a body moving under no forces about a fixed point is expressible in terms of elliptic functions by the relation

$$\omega=(\phi_1+x)^{\frac{1}{2}}\alpha\ \text{ where }\ \dot{x}=\sqrt{(4x^3-Ix-J)}\ \text{ and }\ \phi\alpha+V\alpha\phi\alpha=0,$$

α being a constant imaginary vector, ϕ_1 being a linear function coaxial with ϕ and having for its latent cubic $4g^3-Ig-J=0$.

[Compare Art. 84, p. 124. Here the assumed expression for ω gives

$$\tfrac{1}{2}\dot{x}(\phi_1+x)^{-\frac{1}{2}}\alpha+V(\phi_1+x)^{\frac{1}{2}}\alpha\phi(\phi_1+x)^{\frac{1}{2}}\alpha=0$$

or $$\tfrac{1}{2}\dot{x}(\phi_1+x)^{-\frac{1}{2}}\alpha+(\phi_1+x)^{-\frac{1}{2}}V\alpha\phi\alpha\sqrt{m_1(x)}=0,$$

where $m_1(x)$ is the third invariant of ϕ_1+x. We may obviously take the first invariant of ϕ_1 to be zero without loss of generality, so that the latent cubic of the specified type, and the differential equation for x is reduced to Weierstrass's standard form.

The function ϕ_1 is of the form $\phi_1=a+b\chi+c\psi$ where χ and ψ are the auxiliary functions for ϕ, and when the first invariant is taken to be zero, $3a+2bm''+cm'=0$. The scalars b and c are arbitrary constants of integration. Assuming $\alpha=ui+vj+wk$ where i, j, k are the axes of ϕ, we see that $Au=(B-C)vw$, $Bv=(C-A)wu$, $Cw=(A-B)uv$, A, B and C being the latent roots of ϕ—the principal moments of inertia. Thus

$$\alpha=i\sqrt{\frac{BC}{(C-A)(A-B)}}+j\sqrt{\frac{CA}{(A-B)(B-C)}}+k\sqrt{\frac{AB}{(B-C)(C-A)}},$$

and the latent roots of ϕ_1 are $\frac{1}{3}b(B+C-2A)-\frac{1}{3}c(CA+AB-2BC)$, etc.

Moreover since by definition of α, we have $S\alpha\phi\alpha=0$, $S\alpha\phi^2\alpha=0$ and also $\alpha^2=+1$ as may be easily shown, we find $S\omega\phi\omega=S\alpha\phi(\phi_1+x)\alpha=cABC$ and $(\phi\omega)^2=S\alpha\phi^2(\phi_1+x)\alpha=-bABC$, or in the notation of the last example, $cABC=-h^2$ and $bABC=g^2$.]

Ex. 6. Resolve the vector of angular momentum $\phi\omega$, along and at right angles to ω, and investigate the relation of the components to the quadric $S\varpi\phi\varpi=-1$.

[Compare Art. 111, p. 181.]

Ex. 7. The motion of a freely moving body is known, and it is required to determine as far as possible its dynamical constants.

[The mass cannot be determined, but if we know the particulars of the motion of three points, the extremities of ϵ_1, ϵ_2 and ϵ_3, we can find ω from the two equations $\dot\epsilon_1-\dot\epsilon_2=V\omega(\epsilon_1-\epsilon_2)$, $\dot\epsilon_1-\dot\epsilon_3=V\omega(\epsilon_1-\epsilon_3)$.

In the next place, to find ρ, the vector to the centre of mass, we have

$$\dot\epsilon_1=\dot\rho+V\omega(\epsilon_1-\rho),\text{ and }\ddot\epsilon_1=\ddot\rho+V\dot\omega(\epsilon_1-\rho)+V\omega V\omega(\epsilon_1-\rho),$$

and because $\ddot\rho=0$ the second of these relations gives ρ on solution of a linear equation. To find the function ϕ corresponding to the centre of mass, we differentiate ω twice and use the results of Ex. 4, (f) and (g).]

Ex. 8. Given four particles whose united mass is that of a given rigid body, it is required to connect the particles by a light frame-work, so that the dynamical constants of the system may be identical with those of the body.

[If $\phi\lambda=-\Sigma m_1\rho_1 S\rho_1\lambda$ where λ is an arbitrary vector, and where the vectors ρ_1, etc., are drawn from the centre of mass of the body to terminate at the particles of mass m_1, etc., the problem is solved when we reduce the function ϕ to the form

$$\phi\lambda=-a\alpha S\alpha\lambda-b\beta S\beta\lambda-c\gamma S\gamma\lambda-d\delta S\delta\lambda\text{ where }a\alpha+b\beta+c\gamma+d\delta=0,$$

a, b, c and d being the masses of the four particles and α, β, γ and δ being their vectors of position. Now for some scalar x, we have

$$xa=S\beta\gamma\delta,\quad xb=-S\alpha\gamma\delta,\quad xc=S\alpha\beta\delta,\quad xd=-S\alpha\beta\gamma\,;$$

and we also have (Art. 65, Ex. 1, p. 92)

$$\psi\lambda=-\Sigma abV\alpha\beta S\alpha\beta\lambda,\quad m=\Sigma abcS\alpha\beta\gamma^2.$$

The second of these serves to determine x, for it reduces to $m=x^2abcd\Sigma a$. Substituting α for λ in the first, we find $\psi\alpha=xbcdV(\beta-\delta)(\gamma-\delta)$; and when we operate with $S\alpha$, $S\beta$ and $S\gamma$ on this and similar expressions we have $S\alpha\psi\alpha=-x^2bcd(b+c+d)$, etc., $S\alpha\psi\beta=\text{etc.}=x^2abcd$. It easily appears that the six relations in α, β and γ imply the remaining six involving δ when $\Sigma a\alpha=0$.

Assuming first any vector a which satisfies the condition

$$Sa\psi a = -x^2bcd(b+c+d)$$

—that is any vector which terminates on a certain quadric—we have next the two relations $Sa\psi\beta = x^2abcd$, $S\beta\psi\beta = -x^2acd(a+c+d)$ which require the vector β to terminate on a conic. Selecting β there remain the three equations $Sa\psi\gamma = S\beta\psi\gamma = x^2abcd$, $S\gamma\psi\gamma = -x^2abd(a+b+d)$ which determine γ as the vector to a point of intersection of a line and a quadric. Finally, we have $\delta = -d^{-1}(aa+b\beta+c\gamma)$.]

ART. **122.** When an impulse acts on a system of particles, the velocity of the particle m_1 is changed from $\dot\rho_{1,0}$ to $\dot\rho_1$ where

$$m_1(\dot\rho_1 - \dot\rho_{1,0}) = \lambda_1 + \lambda_{12} + \lambda_{13} + \text{etc.}, \quad \ldots\ldots\ldots\ldots\ldots(\text{I.})$$

where λ_1 is the external impulse acting on m_1 and where λ_{12}, λ_{13}, etc., are the impulsive actions of the particles m_2, m_3, etc., on m_1. These impulsive interactions satisfy conditions analogous to (II.) of Art. 119, and we obtain the equations

$$\Sigma m_1(\dot\rho_1 - \dot\rho_{1,0}) = \Sigma\lambda_1, \quad \Sigma m_1 V\rho_1(\dot\rho_1 - \dot\rho_{1,0}) = \Sigma V\rho_1\lambda_1, \quad \ldots\ldots(\text{II.})$$

which are independent of the interactions. The work done on the particle m_1 by the impulse is (Thomson and Tait, Art. 308)

$$-\tfrac{1}{2}S(\dot\rho_1 + \dot\rho_{1,0})(\lambda_1 + \lambda_{12} + \lambda_{13} + \text{etc.}), \quad \ldots\ldots\ldots\ldots\ldots(\text{III.})$$

and the total work done on the whole system is

$$W = -\tfrac{1}{2}\Sigma S(\dot\rho_1 + \dot\rho_{1,0})\lambda_1 - \tfrac{1}{2}\Sigma S(\dot\rho_1 - \dot\rho_2 + \dot\rho_{1,0} - \dot\rho_{2,0})\lambda_{12}. \quad \ldots(\text{IV.})$$

For a rigid body it is frequently convenient to define the motion by the velocity (σ) of the point of the body coinciding with the origin and the angular velocity (ω). Thus $\dot\rho_1 = \sigma - V\rho_1\omega$, and if

$$\lambda = \Sigma\lambda_1, \quad \mu = \Sigma V\rho_1\lambda_1, \quad \phi\omega = \Sigma V\rho_1 V\omega\rho_1, \quad \ldots\ldots\ldots\ldots(\text{V.})$$

so that λ is the resultant force and μ the resultant moment of the impulse with respect to the origin while ϕ is the inertia function corresponding to the origin, the equations (II.) become

$$M(\sigma - \sigma_0 - V\rho(\omega - \omega_0)) = \lambda, \quad MV\rho(\sigma - \sigma_0) + \phi(\omega - \omega_0) = \mu\,; \quad (\text{VI.})$$

and because λ_{12} is parallel to the line joining two particles and therefore perpendicular to $\dot\rho_1 - \dot\rho_2$ and to $\dot\rho_{1,0} - \dot\rho_{2,0}$, the expression for the work done is independent of λ_{12}, etc., and reduces to

$$W = -\tfrac{1}{2}S(\sigma + \sigma_0)\lambda - \tfrac{1}{2}S(\omega + \omega_0)\mu, \quad \ldots\ldots\ldots\ldots\ldots(\text{VII.})$$

because we have

$$\Sigma S(\sigma + \sigma_0 - V\rho_1(\omega + \omega_0))\lambda_1 = S(\sigma + \sigma_0)\Sigma\lambda_1 + S(\omega + \omega_0)\Sigma V\rho_1\lambda_1.$$

When the origin is taken at the centre of mass, (VI.) becomes

$$M(\sigma - \sigma_0) = \lambda, \quad \phi_0(\omega - \omega_0) = \mu, \quad \ldots\ldots\ldots\ldots\ldots(\text{VIII.})$$

where ϕ_0 refers to the centre of mass, and thus we have at once

$$\sigma = \sigma_0 + M^{-1}\lambda, \quad \omega = \omega_0 + \phi_0^{-1}\mu\,; \quad \ldots\ldots\ldots\ldots\ldots(\text{IX.})$$

or, in the language of the theory of screws, when a free body having an instantaneous twist velocity (σ_0, ω_0) is acted on by an impulsive wrench (μ, λ), the instantaneous twist velocity immediately after the impulse is $(\sigma_0 + M^{-1}\lambda, \ \omega_0 + \phi_0{}^{-1}\mu)$, the centre of mass being the base-point. (See p. 171.)

When the origin is taken at an arbitrary point, we may replace (VI.) by

$$M(\sigma - \sigma_0 - \mathrm{V}\rho(\omega - \omega_0)) = \lambda, \quad \phi_0(\omega - \omega_0) = \mu - \mathrm{V}\rho\lambda, \ \ldots\ldots(\text{X.})$$

where (σ, ω), (σ_0, ω_0) and (μ, λ) are referred to the origin as base-point and where ϕ_0 corresponds to the centre of mass. This is easily shown in various ways.

The form of the expression (VII.) is independent of the choice of base-point. In particular when the base-point is at the centre of mass, we find from (VII.), (VIII.) and (IX.),

$$\begin{aligned} W &= -\tfrac{1}{2}(M\sigma^2 + \mathrm{S}\omega\phi_0\omega) + \tfrac{1}{2}(M\sigma_0{}^2 + \mathrm{S}\omega_0\phi_0\omega_0) \\ &= -\tfrac{1}{2}(M^{-1}\lambda^2 + \mathrm{S}\mu\phi_0{}^{-1}\mu) - \mathrm{S}(\sigma_0\lambda + \omega_0\mu). \ \ldots\ldots\ldots(\text{XI.}) \end{aligned}$$

Ex. 1. Prove that the solution of (VI.) is

$$(\omega - \omega_0)(m + M\mathrm{S}\rho\phi\chi\rho + M^2\mathrm{S}\rho\phi\rho \,.\, \rho^2)$$
$$= (\psi + M\chi\rho\mathrm{S}\rho + M\mathrm{V}\phi\rho\mathrm{V}\rho + M^2\rho^3\mathrm{S}\rho)(\mu - \mathrm{V}\rho\lambda) \,; \quad M(\sigma - \sigma_0) = \lambda + M\mathrm{V}\rho(\omega - \omega_0),$$

where m is the third invariant of ϕ and where χ and ψ are the auxiliary functions.

[Compare Ex. 5, Chap. VIII., p. 102.]

Ex. 2. A rigid body is moving in any manner and an impulsive force is applied to a given point of the body so as to cause that point to move instantaneously with a given velocity. Determine all particulars.

[The centre of mass being taken as base-point, and α being the vector to the point in question and $\dot{\alpha}$ being the velocity of the point, the equations

$$M(\sigma - \sigma_0) = \lambda, \quad \phi(\omega - \omega_0) = \mathrm{V}\alpha\lambda, \quad \sigma - \mathrm{V}\alpha\omega = \dot{\alpha}$$

serve to determine the unknowns σ, ω and λ. We have on elimination of σ and λ, $\phi\omega - M\alpha\mathrm{V}\alpha\omega = \phi\omega_0 + M\mathrm{V}\alpha(\dot{\alpha} - \sigma_0)$; and by Ex. 5, Chap. VIII., the solution may be written

$$(\omega - \omega_0)(m - M\mathrm{S}\alpha\phi\chi\alpha + M^2\alpha^2\mathrm{S}\alpha\phi\alpha) = M\psi\mathrm{V}\alpha(\dot{\alpha} - \dot{\alpha}_0) - M^2\mathrm{V}\phi\alpha\mathrm{V}\alpha\mathrm{V}\alpha(\dot{\alpha} - \dot{\alpha}_0),$$

where $\dot{\alpha}_0 = \sigma_0 - \mathrm{V}\alpha\omega_0$ is the initial velocity of the point. Hence in terms of $\omega - \omega_0$ as given by this equation

$$\sigma - \sigma_0 = \dot{\alpha} - \dot{\alpha}_0 + \mathrm{V}\alpha(\omega - \omega_0) \quad \text{and} \quad \lambda = M^{-1}(\sigma - \sigma_0).]$$

Ex. 3. A rigid body is moving in any manner. Suddenly a line in the body is constrained to move in a definite manner.

[If α and β are the vectors from the centre of mass to two points on the line, we may suppose the impulsive wrench to consist of forces λ and λ' applied at the extremities of α and β. Hence

$$M(\sigma - \sigma_0) = \lambda + \lambda', \quad \phi(\omega - \omega_0) = \mathrm{V}(\alpha\lambda + \beta\lambda'), \quad \sigma = \dot{\alpha} + \mathrm{V}\alpha\omega = \dot{\beta} + \mathrm{V}\beta\omega,$$

where $\dot{\alpha}$ and $\dot{\beta}$ are the velocities of the extremities of α and β. From the first and second equation we deduce $\mathrm{S}(\beta - \alpha)\phi(\omega - \omega_0) + M\mathrm{S}\alpha\beta(\sigma - \sigma_0) = 0$, which asserts that the moment of momentum about the line is unchanged.

We also have $\omega=(\dot{\alpha}-\dot{\beta}+x)(\alpha-\beta)^{-1}$, where x is a scalar to be determined by substituting for ω and σ in the equation just found. Solving the linear equation for x we find ω, and hence σ and λ and λ'.]

Ex. 4. A rigid body is moving in any manner. It is required with the least possible expenditure of energy to cause a given point to move in a given manner.

[Writing equation (XI.) in the form

$$W=-\tfrac{1}{2}(M(\dot{\alpha}+\mathrm{V}\alpha\omega)^2+\mathrm{S}\omega\phi\omega)+\tfrac{1}{2}(M(\dot{\alpha}_0+\mathrm{V}\alpha\omega_0)^2+\mathrm{S}\omega_0\phi\omega_0),$$

we express that this function of ω is a minimum. We find

$$\phi\omega-M\alpha\mathrm{V}\alpha\omega=M\mathrm{V}\alpha\dot{\alpha},$$

and as in Ex. 2, this gives

$$\omega(m-M\mathrm{S}\alpha\phi\chi\alpha+M^2\alpha^2\mathrm{S}\alpha\phi\alpha)=M\psi\mathrm{V}\alpha\dot{\alpha}-M^2\mathrm{V}\phi\alpha\mathrm{V}\alpha\mathrm{V}\alpha\dot{\alpha},$$

and substituting in $\sigma=\dot{\alpha}+\mathrm{V}\alpha\omega$, in $\phi(\omega-\omega_0)=\mu$ and in $M(\sigma-\sigma_0)=\lambda$, we determine the impulsive wrench and the instantaneous twist velocity.]

Ex. 5. If p and p' are the pitches of the screws of an impulsive wrench and of the instantaneous twist velocity produced by the wrench on a free quiescent rigid body; if also ϖ and ϖ' are the vector perpendiculars from the centres of mass on the axes of these screws, $M\sigma=M(p'+\varpi')\omega=\lambda$, $\mu=(p+\varpi)\lambda=\phi\omega$.

(a) Hence in terms of λ and ω,

$$p'=M^{-1}\mathrm{S}\lambda\omega^{-1},\ \ \varpi'=M^{-1}\mathrm{V}\lambda\omega^{-1};\ \ p=\mathrm{S}\phi\omega\lambda^{-1},\ \ \varpi=\mathrm{V}\phi\omega\lambda^{-1};$$
$$M\mathrm{T}\omega\sqrt{(p'^2+\mathrm{T}\varpi'^2)}=\mathrm{T}\lambda,\ \ \mathrm{T}\lambda\sqrt{(p^2+\mathrm{T}\omega^2)}=\mathrm{T}\phi\omega;$$
$$\mathrm{T}\phi\mathrm{U}\omega=M\sqrt{(p^2+\mathrm{T}\varpi^2)}\sqrt{(p'^2+\mathrm{T}\varpi'^2)}.$$

(b) The shortest vector from the axis of the impulsive screw to that of the instantaneous screw is $\varpi'(\mathrm{S}\varpi\varpi'^{-1}-1)$.

(c) Show that

$$\phi\omega\,.\,\omega^{-1}=M(p+\varpi)(p'+\varpi');\quad \lambda=M(p'+\varpi')\phi^{-1}(p+\varpi)\lambda;$$

and express the moment of inertia about the line through the centre of mass parallel to the instantaneous axis in terms of p, p', ϖ and ϖ'.

(d) The cosine of the angle between the axes of the two screws is $p'(p'^2+\mathrm{T}\varpi'^2)^{-\frac{1}{2}}$; and if the axes are parallel, that of the instantaneous screw passes through the centre of mass or else the instantaneous motion is a translation. In the former case the pitch and vector perpendicular on the axis of the impulsive screw satisfy the condition

$$\varpi\mathrm{S}\phi\lambda\lambda^{-1}=p\mathrm{V}\phi\lambda\lambda^{-1}.$$

Ex. 6. Determine the dynamical constants of a free body by observing the effects produced by impulsive wrenches in starting the body from a given position.

[If ρ is the vector from a fixed origin to the unknown centre of mass, if an impulsive wrench is (μ,λ) and the corresponding twist velocity is (σ,ω) for the fixed origin as base-point, the equations are (compare (x.))

$$M(\sigma-\mathrm{V}\rho\omega)=\lambda,\quad \phi\omega=\mu-\mathrm{V}\rho\lambda,$$

together with others with accented letters σ', ω', μ', λ', σ'', ω'', μ'', λ'' for other impulsive wrenches and the corresponding twist velocities. From these equations M, ρ and ϕ (corresponding to the centre of mass) are to be determined. The mass follows at once from the first equation, and we have

$$M=\mathrm{S}\lambda\omega(\mathrm{S}\sigma\omega)^{-1}=\mathrm{S}\lambda'\omega'(\mathrm{S}\sigma'\omega')^{-1}=\mathrm{S}\lambda''\omega''(\mathrm{S}\sigma''\omega'')^{-1}.$$

The vector ρ is given by

$$V(\sigma - M^{-1}\lambda)(\sigma' - M^{-1}\lambda') = VV\rho\omega V\rho\omega' = -\rho S(\sigma - M^{-1}\lambda)\omega'.$$

And the function ϕ can be found from three couple equations. Some rather elegant identities connecting the wrenches and the twist velocities may be deduced from this beautiful problem of Sir Robert Ball's.]

Ex. 7. An impulsive wrench of given pitch and intensity is applied to a free quiescent rigid body. The axis of the screw of the wrench passes through a fixed point; find the direction of the axis so that (a) the kinetic energy, or (b) the angular velocity, generated by the impulse may be as great as possible.

[The base-point being taken at the centre of mass, we have $M.\sigma = \lambda$, $\phi\omega = (p + V\gamma)\lambda$ where $T\lambda$, p and γ are given. The kinetic energy is $-\frac{1}{2}S(p+V\gamma)\lambda\phi^{-1}(p+V\gamma)\lambda - \frac{1}{2}M^{-1}\lambda^2$, and if this is a maximum subject to the condition that $T\lambda$ is given, we have $(p - V\gamma)\phi^{-1}(p+V\gamma)\lambda = g\lambda$ where g is a scalar—a latent root of the self-conjugate function on the left, and for a maximum g is the greatest latent root. The kinetic energy is

$$\tfrac{1}{2}(g + M^{-1})T\lambda^2.$$

The least latent root answers to minimum kinetic energy. For a maximum or minimum angular velocity deal similarly with the equation

$$(p - V\gamma)\phi^{-2}(p + V\gamma)\lambda = g'\lambda.]$$

Ex. 8. An impulsive wrench (μ, λ) is applied to a free rigid body moving with the instantaneous twist-velocity (σ, ω). The change in the kinetic energy is

$$T - S(\omega\mu + \sigma\lambda),$$

where T is the kinetic energy that would have been generated were the body at rest.

(a) With the same meaning for T, show that the wrench

$$(\mu, \lambda).S(\omega\mu + \sigma\lambda).T^{-1}$$

on the arbitrary screw (μ, λ) leaves the kinetic energy of the body unchanged.

(b) The centre of mass being base-point, any wrench on the screw ($\phi\omega$, $M\sigma$), acting on the body when moving with the twist-velocity (σ, ω), leaves the *screw* of the instantaneous twist-velocity unchanged.

Ex. 9. Two bodies collide. Assuming that the impulsive interaction up to a certain stage of the impact is equivalent to a single force (λ) at the point of contact, the equations of motion are

$$M_1(\sigma_1' - \sigma_1) = \lambda,\ \phi_1(\omega_1' - \omega_1) = Va_1\lambda;\ M_2(\sigma_2' - \sigma_2) = -\lambda,\ \phi_2(\omega_2' - \omega_2) = -Va_2\lambda,$$

where (σ_1, ω_1) and (σ_1', ω_1') are the twist-velocities of the body M_1 just before the commencement of the impact and at the particular stage of the impact under consideration, the centre of mass of M_1 being base-point; where ϕ_1 is the inertia-function of M_1 corresponding to its centre of mass, and where a_1 is the vector from the same origin to the point of contact; σ_2, ω_2, σ_2', ω_2', ϕ_2 and a_2 being in like manner related to the body M_2 and to its centre of mass.

(a) The relative velocity of the points of the bodies in contact is

$$\sigma_1' + V\omega_1'a_1 - \sigma_2' - V\omega_2'a_2 = \sigma_1 + V\omega_1a_1 - \sigma_2 - V\omega_2a_2 + (M_1^{-1} + M_2^{-1})\lambda$$
$$+ V.\phi_1^{-1}Va_1\lambda.a_1 + V.\phi_2^{-1}Va_2\lambda.a_2;$$

or briefly, it is

$$\tau' = \tau + \Phi\lambda,$$

where Φ is a certain self-conjugate function determined by the circumstances of the impact and where τ is the initial relative velocity of the points in contact.

(*b*) For perfectly smooth bodies, $V\lambda\nu=0$, where ν is the normal to the bodies at the point of contact, and the value of λ corresponding to the end of the "first period" of impact is

$$\lambda = -\nu S\nu\tau(S\nu\Phi\nu)^{-1};$$

and the twist-velocity of the body M_1 immediately after the impact is

$$(\sigma_1 - (1+e)M_1^{-1}\nu S\nu\tau(S\nu\Phi\nu)^{-1},\quad \omega_1 - (1+e)\phi_1^{-1}V\alpha_1\nu S\nu\tau(S\nu\Phi\nu)^{-1}),$$

where e is the coefficient of restitution.

(*c*) The total loss of kinetic energy is

$$-(1-e^2)S\tau\nu^2 . (S\nu\Phi\nu)^{-1}.$$

(*d*) For perfectly rough bodies, $V\tau'\nu=0$. The value of λ corresponding to the end of the first period of impact is $\lambda = -\Phi^{-1}\tau$, and the twist-velocity immediately after the impact is

$$(\sigma_1 - (1+e)M_1^{-1}\Phi^{-1}\tau,\quad \omega_1 - (1+e)\phi_1^{-1}V\alpha_1\Phi^{-1}\tau).$$

(*e*) For perfectly rough bodies, the loss of kinetic energy is

$$-(1-e^2)S\tau\Phi^{-1}\tau.$$

ART. **123.** When a rigid body is not perfectly free but constrained in any manner an impulsive wrench will in general be partially neutralized by the reaction of the constraints. Referred to the centre of mass as base-point, we have for a quiescent body,

$$M\sigma = \lambda - \lambda_{,},\quad \phi\omega = \mu - \mu_{,}, \quad \text{.....................(I.)}$$

where (μ, λ) is the impulsive wrench and $(\mu_{,}, \lambda_{,})$ the wrench on the constraints, or where $(-\mu_{,}, -\lambda_{,})$ is the reaction of the constraints. In order to determine the instantaneous motion produced by the impulsive wrench (μ, λ), it is necessary to know the *evoked wrench* $(\mu_{,}, \lambda_{,})$. We consider the case in which the constraints are smooth, or so that no evoked wrench can generate any motion. In this case the work done by the wrench $(\mu_{,}, \lambda_{,})$ must be zero, or we must have (Art. 122 (VII.))

$$S(\mu_{,}\omega + \lambda_{,}\sigma) = 0, \quad \text{...........................(II.)}$$

where $(\mu_{,}, \lambda_{,})$ is any wrench arising from the constraints and where (σ, ω) is any possible twist velocity of the body. The screws of $(\mu_{,}, \lambda_{,})$ and of (σ, ω) are said to be *reciprocal* when this condition is satisfied; and for smooth constraints, *every possible twist velocity is reciprocal to every possible wrench arising from the constraints.*

A body with one degree of freedom can move only one way from a given position, by a twist about some definite screw (σ_1, ω_1). A body with two degrees of freedom can move in a singly infinite variety of ways from a given position; if (σ_1, ω_1) and (σ_2, ω_2) are two screws about which the body can begin to twist, it can begin to twist about every screw of the *two-system* $(x_1\sigma_1+x_2\sigma_2,\ x_1\omega_1+x_2\omega_2)$, where x_1 and x_2 are scalars, as easily appears from the composition of small displacements $(\sigma_1 dt_1,\ \omega_1 dt_1)$ and $(\sigma_2 dt_2,\ \omega_2 dt_2)$. Similarly a body with n degrees of freedom can begin to twist about any screw of the n-system $(\Sigma x_1\sigma_1,\ \Sigma x_1\omega_1)$, where $(\sigma_1, \omega_1)\ldots(\sigma_n, \omega_n)$ are n independent screws about which the body can begin to twist; and being given n independent screws about which the body can begin to twist, all possible initial motions belong to a given system of twists. Every wrench reciprocal to n independent screws of the freedom is a wrench arising from the constraints, for every such wrench is reciprocal to every possible twist on account of the linear character of the condition of reciprocity (II.), and no such wrench can generate any motion in the body. By expressing that a wrench $(\mu_,, \lambda_,)$ is reciprocal to n screws of the freedom, the number of its arbitrary constants is reduced from 6 to $6-n$ since n conditions (II.) must be satisfied; and thus the screws of the constraint compose a system of order $(6-n)$. This system can be determined when the system of the freedom is known, and conversely.

Again knowing the system of screws of the freedom we can determine what Sir Robert Ball calls the screws of the *reduced wrenches.* A reduced wrench causes no reaction on the constraints; it produces the same initial motion as if the body were perfectly free. In equations (I.) the wrench $(\mu-\mu_,, \lambda-\lambda_,)$ is a reduced wrench, or $(\phi\omega, M\sigma)$ is the reduced wrench corresponding to the twist velocity (σ, ω). The system of screws of the reduced wrenches is $(\phi\Sigma x_1\omega_1,\ M\Sigma x_1\sigma_1)$ when that of the freedom is $(\Sigma x_1\sigma_1,\ \Sigma x_1\omega_1)$.

Suppose now that we select n independent screws of the n-system of the reduced wrenches and $6-n$ screws of the $(6-n)$-system of the constraints, and that (Art. 102) we resolve an impulsive wrench (μ, λ) into its components on these six screws, we shall have (compare (XVI.), p. 166),

$$\mu=\mu'+\mu_,,\quad \lambda=\lambda'+\lambda_,, \qquad \text{(III.)}$$

where (μ', λ') is the component of (μ, λ) belonging to the system of the reduced wrenches and where $(\mu_,, \lambda_,)$ is the component belonging to the system of the wrenches of the constraint. The instantaneous twist velocity is then given by the relations

$$\sigma=M^{-1}\lambda',\quad \omega=\phi^{-1}\mu'. \qquad \text{(IV.)}$$

Ex. 1. Prove that

$$\mu=\phi\lambda,\quad \mu'=-\phi'\lambda'$$

represent respectively a three-system of screws (μ, λ) and the reciprocal three-system (μ', λ'), ϕ being a given linear vector function and λ and λ' being arbitrary vectors.

[Compare Art. 102 (IV.), and observe that if (μ', λ') is reciprocal to every screw of the system (μ, λ), we must have $S(\mu\lambda'+\mu'\lambda)=0$ or $S\lambda(\phi'\lambda'+\mu')=0$ for all vectors λ.]

Ex. 2. Determine triads of co-reciprocal screws of a three-system.

[If the screws (μ_1, λ_1), (μ_2, λ_2) and (μ_3, λ_3) of the system $\mu=\phi\lambda$ are mutually reciprocal, $S\lambda_1(\phi+\phi')\lambda_2=0$, $S\lambda_2(\phi+\phi')\lambda_3=0$, $S\lambda_3(\phi+\phi')\lambda_1=0$; or λ_1, λ_2 and λ_3 are parallel to mutually conjugate radii of the quadric $S\rho(\phi+\phi')\rho=\text{const.}$ Thus

$$\lambda_1 \parallel (\phi+\phi')^{-\frac{1}{2}}i, \quad \lambda_2 \parallel (\phi+\phi')^{-\frac{1}{2}}j, \quad \lambda_3 \parallel (\phi+\phi')^{-\frac{1}{2}}k,$$

where i, j and k are three mutually perpendicular unit vectors.]

Ex. 3. Determine sextets of co-reciprocal screws.

[Take any triad of co-reciprocal screws of a three-system $\mu=\phi\lambda$, and any triad of co-reciprocal screws of the reciprocal system $\mu=-\phi'\lambda$.]

Ex. 4. Resolve a wrench (or twist) into its components on six co-reciprocal screws.

[If $(\mu_1, \lambda_1)\ldots(\mu_6, \lambda_6)$ are the six co-reciprocals, we can find a linear function ϕ so that (μ_1, λ_1), (μ_2, λ_2) and (μ_3, λ_3) belong to the system $\mu'=\phi\lambda'$; and then (μ_4, λ_4), (μ_5, λ_5) and (μ_6, λ_6) will belong to the system $\mu''=-\phi'\lambda''$. We assume for the given wrench (μ, λ) that $\mu=\phi\lambda'-\phi'\lambda''$ and $\lambda=\lambda'+\lambda''$; whence we have generally $\lambda'=(\phi+\phi')^{-1}(\mu+\phi'\lambda)$ and $\lambda''=-(\phi+\phi')^{-1}(\mu-\phi\lambda)$, and it only remains to resolve λ' along λ_1, λ_2 and λ_3, and λ'' along λ_4, λ_5 and λ_6 in order to obtain the required relations $\mu=\Sigma x_1\mu_1$, $\lambda=\Sigma x_1\lambda_1$.]

Ex. 5. Find the $(n-6)$-system reciprocal to a given n-system.

[This has been effected in Ex. 1 for $n=3$. Let $n=4$, and for any three screws of the system construct the function ϕ. Resolve any fourth screw (μ_n, λ_n) as in the last example, so that $\mu_n=\phi\lambda'-\phi'\lambda''$ and $\lambda_n=\lambda'+\lambda''$, and take two vectors λ_5 and λ_6 which with λ'' compose a mutually conjugate triad with respect to $S\rho(\phi+\phi')\rho=\text{const.}$ Then

$$(-x_5\phi'\lambda_5-x_6\phi'\lambda_6, \quad x_5\lambda_5+x_6\lambda_6)$$

is the two-system reciprocal to the four-system. To determine the four-system reciprocal to a given two-system, take any function ϕ satisfying $\mu_1=\phi\lambda_1$, $\mu_2=\phi\lambda_2$, where (μ_1, λ_1) and (μ_2, λ_2) are two screws of the two-system, and determine the vector λ_3 conjugate to λ_1 and λ_3 with respect to the quadric $S\rho(\phi+\phi')\rho=\text{const.}$ The required four-system is

$$(x_3\phi\lambda_3-\phi'\lambda', \quad x_3\lambda_3+\lambda'),$$

where λ' and x_3 are arbitrary. Similarly we may proceed in other cases.]

Ex. 6. Show that

$$S(\mu\lambda'+\mu'\lambda)=(p+p')S\lambda\lambda'+S(\gamma-\gamma')\lambda\lambda',$$

where p and p' are the pitches of the screws (μ, λ) and (μ', λ'), and where γ and γ' are the vectors to points on their axes. Interpret this result, and show that

$$-S(\mu\lambda'+\mu'\lambda)=(p+p')\cos u+d\sin u,$$

where u is the angle and where d is the shortest distance between the axes.

Ex. 7. A body twisting along the screw (σ_1, ω_1) is suddenly constrained to twist along another screw (σ_2, ω_2). Determine the motion.

[If $(x\sigma_1, x\omega_1)$ is the twist velocity just before the change and $(y\sigma_2, y\omega_2)$ that just after, we have

$$M(y(\sigma_2-V\rho\omega_2)-x(\sigma_1-V\rho\omega_1))=\lambda, \quad y\phi\omega_2-x\phi\omega_1=\mu-V\rho\lambda,$$

where (μ, λ) is the wrench arising from the constraint which produces the change of motion. This wrench is reciprocal to (σ_2, ω_2), so that

$$S(\sigma_2\lambda + \omega_2\mu) = 0.$$

Substituting we easily find y to be given in terms of x by the relation

$$y(M(\sigma_2 - V\rho\omega_2)^2 + S\omega_2\phi\omega_2) = x(MS(\sigma_1 - V\rho\omega_1)(\sigma_2 - V\rho\omega_2) + S\omega_1\phi\omega_2).]$$

Ex. 8. A body oscillates under the action of a conservative system of forces, and at a certain part of its swing the motion is suddenly changed from a twist about one given screw (σ_1, ω_1) to a twist about another (σ_2, ω_2). Show that the twist velocities just before the sudden changes of motion at the beginning and end of a complete oscillation are in the ratio

$$(M\sigma_1^2 + S\omega_1\phi\omega_1)(M\sigma_2^2 + S\omega_2\phi\omega_2) : (MS\sigma_1\sigma_2 + S\omega_1\phi\omega_2)^2,$$

the base-point being coincident with the position of the centre of mass at the instant of the change of motion.

[This is the general case of a self-closing gate. By the last example

$$y(M\sigma_2^2 + S\omega_2\phi\omega_2) = x(MS\sigma_1\sigma_2 + S\omega_1\phi\omega_2)$$

and

$$y(MS\sigma_1\sigma_2 + S\omega_1\phi\omega_2) = x'(M\sigma_1^2 + S\omega_1\phi\omega_1),$$

where $x : x'$ is the ratio of the twist velocities just before the change from the screw (σ_1, ω_1) to the screw (σ_2, ω_2) and just after the change from (σ_2, ω_2) back to (σ_1, ω_1). The system of forces being conservative, the magnitude of the twist velocity throughout the partial oscillation during the continuous part of the swing depends solely on the position of the body, and is the same just after the sudden change from (σ_2, ω_2) to (σ_1, ω_1) as just before the next sudden change from (σ_1, ω_1) to (σ_2, ω_2). To show that x is greater than x' or that $(M\sigma_1^2 + S\omega_1\phi\omega_1)(M\sigma_2^2 + S\omega_2\phi\omega_2) - (MS\sigma_1\sigma_2 + S\omega_1\phi\omega_2)^2$ is positive, turns on the fact that $\alpha^2\beta^2 + \gamma^2\delta^2 - 2S\alpha\beta S\gamma\delta$ is positive when α, β, γ and δ are real vectors. The value of this expression lies between the limits $(T\alpha\beta \pm T\gamma\delta)^2$.]

Ex. 9. An impulsive wrench reciprocal to the instantaneous twist velocity of a free body at the moment of its application increases the kinetic energy.

[The change of kinetic energy (Art. 122 (XI.) is $-\frac{1}{2}S\mu\phi^{-1}\mu - \frac{1}{2}M^{-1}\lambda^2$, and this is equal to the kinetic energy which the wrench would generate were the body at rest.]

Ex. 10. Determine the dynamical constants and the constraints of a rigid body by observing the effects of impulsive wrenches applied to the body when placed in a given position.

[Let (μ_1, λ_1), $(\mu_{,1}, \lambda_{,1})$ and (σ_1, ω_1) represent an impulsive wrench, the corresponding opposing wrench arising from the constraints and the twist velocity produced. We know (σ_1, ω_1) by observation—that is, a screw of the freedom. A second impulsive wrench (μ_2, λ_2) being applied, we find a second screw of the freedom (σ_2, ω_2), provided we have not $\sigma_2 = t\sigma_1$, $\omega_2 = t\omega_1$. In this second case, however, we have a screw of the constraint, for the impulsive wrench $(\mu_2 - t\mu_1, \lambda_2 - t\lambda_1)$ generates no motion. Administering a third wrench we obtain similarly either a new screw of the freedom or a new screw of the constraint; and from the results of applying six independent wrenches, the screw systems of the freedom and of the constraint become completely known. These systems being known, we can by (III.) resolve an impulsive wrench (μ_1, λ_1) into the reduced wrench (μ_1', λ_1') and the evoked wrench $(\mu_{,1}, \lambda_{,1})$; and we have as many sets of equations $M(\sigma_1 - V\rho\omega_1) = \lambda_1'$, $\phi\omega_1 = \mu_1' - V\rho\lambda_1'$ as degrees of freedom. For one degree of freedom, the first equation gives the mass $M = S\omega_1\lambda_1' : S\omega_1\sigma_1$; and a line locus

$$V\omega_1(\rho S\omega_1\lambda_1' - V\sigma_1\lambda_1') = 0$$

for the centre of mass. Eliminating ρ between this and the second equation, the result is

$$\mathrm{V}\,.\,\mathrm{V}\omega_1\lambda_1'(\mu_1'-\phi\omega_1)\mathrm{S}\omega_1\lambda_1'=\lambda_1'\mathrm{SV}\omega_1\lambda_1'\mathrm{V}\sigma_1\lambda_1';$$

or separately,

$$\mathrm{S}\lambda_1'\phi\omega_1=\mathrm{S}\lambda_1'\mu_1' \text{ and } \mathrm{S}\omega_1\phi\omega_1=\mathrm{S}(\omega_1\mu_1'+\sigma_1\lambda_1')-\lambda_1'^2\mathrm{S}\sigma_1\omega_1(\mathrm{S}\omega_1\lambda_1')^{-1}.$$

The body has therefore a given moment of inertia $(\mathrm{S}\omega_1^{-1}\phi\omega_1)$ round ω_1, and a given product of inertia $(-\mathrm{SU}\omega_1\phi\mathrm{U}\lambda_1')$ with respect to $\mathrm{U}\omega_1$ and $\mathrm{U}\lambda_1'$; but it is otherwise indeterminate.

For two degrees of freedom, the two force equations completely determine ρ, and the couple equations give completely $\phi\omega_1$ and $\phi\omega_2$. There remains only one unknown constant, the moment of inertia $(\mathrm{SV}\omega_1\omega_2^{-1}\phi\mathrm{V}\omega_1\omega_2)$ with respect to the line perpendicular to ω_1 and ω_2.

The dynamical constants are completely determinate in the case of three degrees of freedom. Compare generally Art. 122, Ex. 6.]

Ex. 11. Two three-systems of screws can be in one way correlated, so that each screw of one system, regarded as an impulsive screw, corresponds to a screw of the other system regarded as an instantaneous screw. (Ball, *Treatise*, Art. 318.)

[This has been virtually proved in the last example. We have to show that if $\sigma=\phi_1\omega$ and $\mu=\phi_2\lambda$ are two three-systems of screws, it is possible to design and place a rigid body so that $M(\sigma-\mathrm{V}\rho\omega)=\lambda$ and $\phi\omega=\mu-\mathrm{V}\rho\lambda$ become identities when σ and μ are replaced in terms of λ and ω and when a one-to-one relation is established between λ and ω. Substituting for μ and σ, we have $M(\phi_1-\mathrm{V}\rho)\omega=\lambda$ and $\phi\omega=(\phi_2-\mathrm{V}\rho)\lambda$, so that

$$\phi\omega=M(\phi_2-\mathrm{V}\rho)(\phi_1-\mathrm{V}\rho)\lambda=M(\phi_1'+\mathrm{V}\rho)(\phi_2'+\mathrm{V}\rho)\lambda,$$

remembering that ϕ is self-conjugate, and this holds for all vectors λ. Hence

$$(\phi_2\phi_1-\phi_1'\phi_2')\lambda-\mathrm{V}\chi_1\rho\lambda-\mathrm{V}\chi_2'\rho\lambda=0,$$

where χ_1 and χ_2 are Hamilton's auxiliary functions for ϕ_1 and ϕ_2. And because λ is perfectly arbitrary, we have $(\chi_1+\chi_2')\rho=2\epsilon_{21}$ if ϵ_{21} is the spin-vector of $\phi_2\phi_1$. Thus the vector to the centre of mass is $2(\chi_1+\chi_2')^{-1}\epsilon_{21}$, and hence $M^{-1}\phi$ is expressed in terms of ϕ_1 and of ϕ_2. The two three-systems are connected by the relation $M(\phi_1-2\mathrm{V}(\chi_1+\chi_2')^{-1}\epsilon_{21})\omega=\lambda$, so that to each screw of one system corresponds a definite screw of the other.]

Ex. 12. Screws (μ, λ) and (σ, ω) are connected by the relations

$$\lambda=\phi_1\sigma+\phi_2\omega, \quad \mu=\phi_3\omega+\phi_4\sigma,$$

where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are four given linear vector functions. Find the conditions that (μ', λ') should be reciprocal to (σ, ω) whenever (μ, λ) is reciprocal to (σ', ω').

[The general relations of this example establish a homography between screws (μ, λ) and (σ, ω); and when the conditions of mutual reciprocity are satisfied, the homography is said to be *chiastic* (Ball).

The conditions are simply

$$\mathrm{S}(\lambda\sigma'+\mu\omega')=\mathrm{S}(\lambda'\sigma+\mu'\omega)$$

or $$\mathrm{S}\sigma'(\phi_1\sigma+\phi_2\omega)+\mathrm{S}\omega'(\phi_3\omega+\phi_4\sigma)=\mathrm{S}\sigma(\phi_1\sigma'+\phi_2\omega')+\mathrm{S}\omega(\phi_3\omega'+\phi_4\sigma'),$$

where ω, ω', σ and σ' are arbitrary vectors. Putting ω and ω' both zero, it appears that ϕ_1 must be self-conjugate. In like manner ϕ_3 is self-conjugate, and the condition reduces to $\mathrm{S}\sigma'(\phi_2-\phi_4')\omega=\mathrm{S}\sigma(\phi_2-\phi_4')\omega'$, which requires ϕ_4 to be the conjugate of ϕ_2. Thus the general chiastic homography is defined by relations of the form

$$\lambda=\phi_1\sigma+\phi_2\omega, \quad \mu=\phi_3\omega+\phi_2'\sigma,$$

where ϕ_1 and ϕ_3 are self-conjugate.]

Ex. 13. The screws of impulsive wrenches applied to a free rigid body at rest in a given position (or the screws of the *reduced* wrenches applied to a constrained body) are in chiastic homography with the screws of the corresponding instantaneous twist velocities.

[Here $\lambda = M(\sigma - V\rho\omega)$, $\mu = \phi\omega + MV\rho(\sigma - V\rho\omega)$ and the conditions are satisfied. This may be seen still more simply by taking the base-point at the centre of mass.]

Ex. 14. The united screws of a chiastic homography are co-reciprocal.

[For a united screw $\mu = x\sigma$, $\lambda = x\omega$, and for a second united screw $\mu' = x'\sigma'$, $\lambda' = x'\omega'$, and hence $xS(\sigma\omega' + \sigma'\omega) = S(\mu\omega' + \sigma'\lambda) = S(\mu'\omega + \sigma\lambda') = x'S(\sigma'\omega + \sigma\omega')$, so that the screws are reciprocal or else $x = x'$. In the latter case every screw of the system $(\sigma + t\sigma', \omega + t\omega')$ is easily seen to be a united screw of the homography. The theory is quite analogous to that of the axes of a self-conjugate function. The united screws in the general homography are to be determined by solution of the equations $x\omega = \phi_1\sigma + \phi_2\omega$, $x\sigma = \phi_3\omega + \phi_4\sigma$. On elimination of σ, we have

$$\phi_3\omega = (\phi_4 - x)\phi_1^{-1}(\phi_2 - x)\omega.$$

Compare Art. 115 (x.), p. 186.]

Ex. 15. There are n real *principal screws* for every position of a rigid body having freedom of the nth order, so that the body will begin to move from rest along one of these screws when a wrench is administered on that screw.

[For the centre of mass as base-point, if (μ, λ) is on a principal screw, we have $\mu = x\sigma$, $\lambda = x\omega$ and also $\mu - \mu_{,} = \phi\omega$ and $\lambda - \lambda_{,} = M\sigma$. Now if (σ_1, ω_1), (σ_2, ω_2), etc., are screws of the freedom we deduce from these expressions the n conditions

$$xS(\sigma_1\omega + \sigma\omega_1) = S\omega\phi\omega_1 + MS\sigma\sigma_1, \text{ etc.};$$

because the evoked wrench is reciprocal to every screw of the freedom. Also $\omega = \Sigma t_n\omega_n$ and $\sigma = \Sigma t_n\sigma_n$, and on substitution for ω and σ and on elimination of the scalars t, a determinant of the nth order in x is obtained. Putting x equal to one of the roots of this equation, the scalars t can be found from $n-1$ of the conditions.

Just as in the case of self conjugate functions, if a root x is imaginary $(x' + \sqrt{-1}x'')$, the corresponding principal screw is imaginary

$$(\sigma' + \sqrt{-1}\sigma'', \lambda' + \sqrt{-1}\lambda'');$$

and there is a conjugate principal screw $(\sigma' - \sqrt{-1}\sigma'', \lambda' - \sqrt{-1}\lambda'')$. By the last example these screws are reciprocal, and we find that

$$S\omega'\phi\omega' + M\sigma'^2 + S\omega''\phi\omega'' + M\sigma''^2$$

must vanish. This cannot be because the energy of a body moving with a real twist-velocity (σ', ω') or (σ'', ω'') is essentially positive.]

Ex. 16. A body which is imperfectly free moves under no applied forces. Find the conditions that the instantaneous screw should be *permanent*.

[When the instantaneous screw is momentarily stationary it is said to be permanent (Sir Robert Ball). For the centre of mass as base-point, the equations of motion are

$$M(\dot\sigma + V\omega\sigma) = -\xi_{,}, \quad \phi\dot\omega + V\omega\phi\omega = -\eta_{,},$$

where $(\eta_{,}, \xi_{,})$ is the evoked wrench. The condition of reciprocity gives $S\omega\phi\dot\omega + MS\sigma\dot\sigma = 0$; and for a permanent screw $\dot\omega = x\omega$, $\dot\sigma = x\sigma$, and we must have $x = 0$ because $S\omega\phi\omega + M\sigma^2$ is essentially negative. By means of the equations of constraint we can eliminate $\xi_{,}$ and $\eta_{,}$ from the conditions

$$MV\omega\sigma = -\xi_{,}, \quad V\omega\phi\omega = -\eta_{,}.]$$

Ex. 17. To find the principal and the permanent screws for freedom of the third order.

[Here $\sigma = \phi_1 \omega$ where ϕ_1 is a given linear function, and the screws of the constraint belong to the reciprocal three-system $\mu_{,} = -\phi_1' \lambda_{,}$. For a principal screw

$$\phi\omega = x\sigma - \mu_{,} = x\phi_1\omega + \phi_1'\lambda_{,}, \quad M\phi_1\omega = x\omega - \lambda_{,};$$

and on elimination of $\lambda_{,}$, we see that

$$(\phi + M\phi_1'\phi_1)\omega = x(\phi_1 + \phi_1')\omega,$$

so that ω is an axis and x a root of a determinate linear vector function.

For a permanent screw,

$$V\omega\phi\omega = -\eta_{,} = \phi_1'\xi_{,}, \quad MV\omega\phi_1\omega = -\xi_{,};$$

and on elimination of $\xi_{,}$ we find

$$V\omega(\phi - M\psi_1)\omega = 0$$

and ω is now an axis of the new linear function $\phi - M\psi_1$.

In the special case of rotation about a fixed point the principal screws coincide with the permanent screws.]

CHAPTER XVI.

THE OPERATOR ∇.

(i) *The Associated Linear Functions.*

ART. **124.** In Articles 54-57 we investigated some fundamental properties of the operator ∇, and we propose in the present chapter to supplement and develop the results already obtained and to illustrate the application of the operator to physical investigations.* Compare pp. 69-77.

In the first place we shall consider the invariants and the auxiliary functions for the linear function

$$\phi a = -\mathrm{S}a\nabla . \sigma, \quad \phi' a = -\nabla\mathrm{S}a\sigma, \qquad \text{(I.)}$$

* Hamilton's writings on the operator ∇ consist, so far as I am aware, and I have searched through his manuscripts in the library of Trinity College, of a communication to the Royal Irish Academy (July 20, 1846) which is published in the *Proceedings*, Vol. iii., p. 291, and practically reprinted in the *Phil. Mag.* of the following year, and of Art. 620 of the *Lectures on Quaternions.* In the *Lectures* he writes: "The bare inspection of these forms may suffice to convince any person who is acquainted, even slightly (and I do not pretend to be well acquainted), with the modern researches in analytical physics, respecting attraction, heat, electricity, magnetism, etc., that the equations of the present article must yet become (as above hinted) extensively useful in the mathematical study of nature, when the calculus of quaternions shall come to attract a more general attention than that which it has hitherto received, and shall be wielded, as an instrument of research by abler hands than mine." He denoted the operator by the symbol $\triangleleft$. In the *Elements* the operator occurs in a disguised form in Art. 418 (v.), ∇ being replaced by $-\mathrm{D}_a$ where a is the vector operand. In the first note to Art. 422 of the same volume and in a letter to Dr. Salmon (Graves's *Life*, Vol. iii., p. 194), he announces his intention of concluding the work with a brief account of a "quaternion transformation of a celebrated equation in partial differential coefficients, of the first order and second degree, which occurs in the theory of heat, and in that of the attraction of spheroids." Unfortunately the volume was left unfinished at his death.

The applications, predicted by Hamilton, have been made by the able hands of Tait, as will be seen on reference to the volumes of his collected *Scientific Papers* (Cambridge, 1900), and to the last edition of his *Treatise on Quaternions* (Cambridge, 1890). M'Aulay has also made valuable additions to the subject in his *Utility of Quaternions in Physics* (Macmillan, 1893), and the note in the Appendix to the new edition of Hamilton's *Elements* (Vol. ii., pp. 432-475) may perhaps be consulted with advantage.

No satisfactory name has been proposed for the operator. The author prefers to call it *Hamilton's delta* or more generally *delta.*

which we noticed in Arts. 112 and 113 in connection with the theory of heterogeneous strain. See p. 181.

When the points of a field receive a small continuous displacement so that the vector to a point changes from ρ to $\rho+\sigma\mathrm{d}t$ where $\mathrm{d}t$ is some small scalar and where σ is a continuous function of ρ, the vector $\rho+\alpha$ to a neighbouring point changes into $\rho+\alpha+(\sigma+\phi\alpha)\mathrm{d}t$. The vector line-element α at the extremity of ρ accordingly changes into $\alpha+\phi\alpha\,.\,\mathrm{d}t$. The vector area-element $\mathrm{V}\alpha\beta$ becomes $\mathrm{V}(\alpha+\phi\alpha\,.\,\mathrm{d}t)(\beta+\phi\beta\,.\,\mathrm{d}t)$, or, neglecting the square of $\mathrm{d}t$, this is (Art. 65 (IV.), p. 91)

$$\mathrm{V}\alpha\beta+\mathrm{V}(\alpha\phi\beta+\phi\alpha\beta)\,.\,\mathrm{d}t \quad \text{or} \quad \mathrm{V}\alpha\beta+\chi'\mathrm{V}\alpha\beta\,.\,\mathrm{d}t.$$

The volume-element $-\mathrm{S}\alpha\beta\gamma$ changes into

$$-\mathrm{S}\alpha\beta\gamma-\Sigma\mathrm{S}\phi\alpha\beta\gamma\,.\,\mathrm{d}t=-\mathrm{S}\alpha\beta\gamma(1+m''\mathrm{d}t)$$

when we neglect the square of $\mathrm{d}t$. If σ denotes the velocity of the points in the field, varying from point to point, and if $\mathrm{d}t$ is the element of time; if $\mathrm{d}\rho$, $\mathrm{d}\nu$ and $\mathrm{d}v$ are respectively a vector line-element, a vector area-element and a volume-element, at the extremity of the vector ρ, the rates at which these elements change are

$$\mathrm{D}_t\,.\,\mathrm{d}\rho=\phi\mathrm{d}\rho, \quad \mathrm{D}_t\,.\,\mathrm{d}\nu=\chi'\mathrm{d}\nu, \quad \mathrm{D}_t\,.\,\mathrm{d}v=m''\mathrm{d}v\,; \;\ldots\ldots\ldots\text{(II.)}$$

and these relations clearly indicate the meanings to be attached in this case to ϕ, χ' and m''. The scalar m'' is called the *divergence* and $\mathrm{S}\nabla\sigma$ is the *convergence.*

Again the small strain at the extremity of ρ due to the displacement $\sigma\mathrm{d}t$ may be resolved into a pure strain, which converts α into $\alpha+\frac{1}{2}(\phi+\phi')\alpha\,.\,\mathrm{d}t$, and a rotation represented in magnitude and direction by $\epsilon\mathrm{d}t$ where ϵ is the spin-vector of ϕ; for we have, if $\phi_0=\frac{1}{2}(\phi+\phi')$,

$$\alpha+\phi\alpha\,.\,\mathrm{d}t=(1+\mathrm{V}\epsilon\,.\,\mathrm{d}t)(\alpha+\phi_0\alpha\,.\,\mathrm{d}t)=(1+\phi_0\mathrm{d}t)(\alpha+\mathrm{V}\epsilon\alpha\,.\,\mathrm{d}t)$$

when we neglect $\mathrm{d}t^2$. Hence the spin-vector ϵ represents in magnitude and direction the *angular velocity* of the element at ρ when σ denotes the velocity of its points in the field.

It remains to exhibit ϵ, m'' and χ in terms of σ. We have for any three vectors

$$\begin{aligned}\mathrm{V}\beta\gamma\,.\,\phi\alpha+\mathrm{V}\gamma\alpha\,.\,\phi\beta+\mathrm{V}\alpha\beta\,.\,\phi\gamma&\\ &=-(\mathrm{V}\beta\gamma\mathrm{S}\alpha\nabla+\mathrm{V}\gamma\alpha\mathrm{S}\beta\nabla+\mathrm{V}\alpha\beta\mathrm{S}\gamma\nabla)\,.\,\sigma\\ &=-\nabla\sigma\mathrm{S}\alpha\beta\gamma=(m''-2\epsilon)\mathrm{S}\alpha\beta\gamma\,;\end{aligned}$$

and the first quaternion invariant (Art. 67, Ex. 7, p. 97) is

$$m''-2\epsilon=-\nabla\sigma, \quad \text{and} \quad m''=-\mathrm{S}\nabla\sigma, \quad \epsilon=\tfrac{1}{2}\mathrm{V}\nabla\sigma. \;\ldots\ldots\text{(III.)}$$

Further, $\quad \chi\alpha=\mathrm{V}\nabla\mathrm{V}\alpha\sigma, \quad \chi'\alpha=-\mathrm{V}\,.\,\mathrm{V}\alpha\nabla\,.\,\sigma, \;\ldots\ldots\ldots\ldots\ldots\text{(IV.)}$

because, for example, $\chi\alpha = (m'' - \phi)\alpha = -S\nabla\sigma . \alpha + S\alpha\nabla . \sigma$. It is evident that χ' and ϕ have the same spin-vector. The vector 2ϵ or $V\nabla\sigma$ has been called by Clerk Maxwell the *curl* of the vector σ.

The function ψ' and the invariants m' and m are related to the transformation which converts vectors ρ into vectors σ where σ is a given but arbitrary function of ρ. As in Art. 63, if $d\sigma$, $d\nu_\sigma (= Vd\sigma d'\sigma)$, and $dv_\sigma (= -Sd\sigma d'\sigma d''\sigma)$, are the elements into which the elements $d\rho$, $d\nu$ and dv at the extremity of ρ transform, we have $d\sigma = \phi d\rho$, $d\nu_\sigma = \psi' d\nu$, $dv_\sigma = m dv$. To calculate ψ in terms of σ it is necessary to use temporary marks to associate the corresponding operator and operand, and we find (p. 90)

$$\psi V\alpha\beta = V\phi'\alpha\phi'\beta = V\nabla S\alpha\sigma . \nabla' S\beta\sigma' = V\nabla\nabla' S\alpha\sigma S\beta\sigma'.$$

Now we may also put

$$\psi V\alpha\beta = V\nabla'\nabla S\alpha\sigma' S\beta\sigma = -V\nabla\nabla' S\alpha\sigma' S\beta\sigma,$$

so that on addition,

$$\psi V\alpha\beta = \tfrac{1}{2} V\nabla\nabla'(S\alpha\sigma S\beta\sigma' - S\alpha\sigma' S\beta\sigma) = -\tfrac{1}{2} V\nabla\nabla' SV\sigma\sigma' V\alpha\beta;$$

or for an arbitrary vector γ,

$$\psi\gamma = -\tfrac{1}{2} V\nabla\nabla' S\sigma\sigma'\gamma, \quad \psi'\gamma = -\tfrac{1}{2} S\gamma\nabla\nabla' . V\sigma\sigma', \quad \text{.......(v.)}$$

and in these expressions the accents are to be removed after the performance of the indicated operations.*

Just as in (III.) we find the quaternion invariant of ψ',

$$m' - 2\phi\epsilon = -\tfrac{1}{2} V\nabla\nabla' V\sigma\sigma', \quad \text{....................(vi.)}$$

remembering that $\phi\epsilon$ is the spin-vector of ψ' (Art. 68, p. 98).

Thus $$m' = -\tfrac{1}{2} SV\nabla\nabla' V\sigma\sigma', \quad \phi\epsilon = \tfrac{1}{4} V . V\nabla\nabla' V\sigma\sigma', \quad \text{........(vii.)}$$

and this expression for $\phi\epsilon$ should be verified by operating with ϕ on the value already obtained for ϵ.

It is also a useful exercise to verify that the third invariant is

$$m = \tfrac{1}{6} S\nabla\nabla'\nabla'' S\sigma\sigma'\sigma'', \quad \text{....................(viii.)}$$

but a more familiar form of this invariant is

$$m = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} \quad \text{.........................(ix.)}$$

*The device employed here is quite analogous to a transformation in Aronhold's symbolic notation.

which is obtained by putting $\rho=ix+jy+kz$, $\sigma=iu+jv+kw$ and

$$m=-S\phi i\phi j\phi k=-S\frac{\partial\sigma}{\partial x}\frac{\partial\sigma}{\partial y}\frac{\partial\sigma}{\partial z}. \quad\text{.................(x.)}$$

Ex. 1. Show that in terms of i, j and k,

$$\phi\alpha=-\Sigma\frac{\partial\sigma}{\partial x}S\alpha i;\quad \psi\alpha=-\Sigma iS\frac{\partial\sigma}{\partial y}\frac{\partial\sigma}{\partial z}\alpha;\quad m''=\Sigma\frac{\partial u}{\partial x},\quad m'=\Sigma\left(\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}-\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}\right).$$

Ex. 2. In terms of three arbitrary differentials of ρ and of the corresponding differentials of σ,

$$\phi\alpha=\frac{\Sigma d\sigma S\alpha d'\rho d''\rho}{Sd\rho d'\rho d''\rho},\quad \psi\alpha=\frac{\Sigma d\rho Sd'\sigma d''\sigma\alpha}{Sd\rho d'\rho d''\rho},\quad m=\frac{Sd\sigma d'\sigma d''\sigma}{Sd\rho d'\rho d''\rho};$$

$$m''+2\epsilon=\frac{\Sigma d\sigma\,.\,Vd'\rho d''\rho}{Sd\rho d'\rho d''\rho},\quad m'-2\phi\epsilon=\frac{\Sigma d\rho\,.\,Vd'\sigma d''\sigma}{Sd\rho d'\rho d''\rho}.$$

(a) If $d\rho=\phi_{\prime}d\sigma$, write down the corresponding functions for $\phi_{\prime}$, and find the relations between them and those for ϕ.

Ex. 3. Prove that

$$\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}\cdot\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}=1.$$

Ex. 4. If σ, a vector function of ρ, satisfies a scalar equation $f(\sigma)=0$ for all values of ρ, the third invariant m of the function ϕ vanishes; and conversely if m vanishes σ satisfies an identical relation.

[If $d\sigma$ is the differential of σ corresponding to an arbitrary differential of ρ, we have $df(\sigma)=0$ or (say) $S\mu d\sigma=0$. Hence the three differentials of σ corresponding to three arbitrary differentials of ρ are coplanar and $Sd\sigma d'\sigma d''\sigma=0$. Conversely, if m is identically zero, three differentials of σ corresponding to three arbitrary differentials of ρ are linearly connected, or $ld\sigma+l'd'\sigma+l''d''\sigma=0$, suppose. Hence σ can receive only two independent variations, or a relation of the form $f(\sigma)=0$ must be satisfied by σ.]

Ex. 5. If σ satisfies two scalar relations $f_1(\sigma)=0$ and $f_2(\sigma)=0$, the function ψ must vanish, and conversely.

Ex. 6. If $f(\sigma)=0$, and if we write $df\sigma=S\mu d\sigma$, we shall have $\phi'\mu=0$.

Ex. 7. If σ is a function of ρ and if $d\sigma=\phi d\rho$, prove by comparing the operators $d=-Sd\rho\nabla=-Sd\sigma\nabla_\sigma$, that

$$\nabla=\phi'\nabla_\sigma,$$

where ∇_σ operates on a function of σ in the same manner as ∇ operates on a function of ρ. (Tait's *Quaternions*, Art. 480.)

Ex. 8. If $\phi d\rho$ is the differential of a vector function of ρ,

$$V\nabla\phi'\alpha=0,$$

where α is an arbitrary constant vector; and if it is possible to find a scalar multiplier to render $\phi d\rho$ the differential of a vector function,

$$S\phi'\alpha\nabla\phi'\alpha=0.$$

[Note that $\phi'\alpha=-\nabla S\alpha\sigma$ if $\phi d\rho=d\sigma$.]

Ex. 9. If C_1 and C_2 are the principal curvatures of a surface $u=$const., show that
$$C_1+C_2=-S\nabla U\nabla u,\quad C_1C_2=-\tfrac{1}{2}SV\nabla\nabla'VU\nabla u\nabla'u'.$$

[See my note, *Elements*, Vol. ii., p. 251. If τ_1 and τ_2 are tangents to the two lines of curvature,
$$C_1\tau_1+S\tau_1\nabla\,.\,U\nabla u=0,\quad C_2\tau_2+S\tau_2\nabla\,.\,U\nabla u=0\,;$$
and (Ex. 4), since $TU\nabla u=1$, the third invariant of the function $-Sd\rho\nabla\,.\,U\nabla u$ is zero, and C_1, C_2 and zero are therefore its latent roots.]

(ii) *Integration Theorems.*

ART. **125**. It has been shown in Arts. 55 and 56 that the form in which the operator ∇ naturally presents itself leads to the two results (pp. 72 and 73).
$$\int d\nu\,.\,q=\nabla q\,.\,dv,\qquad \int d\rho\,.\,q=V(d\nu\,.\,\nabla)\,.\,q\,;\;\ldots\ldots\ldots\ldots(\text{I.})$$
the first integral being taken over a small closed surface of which $d\nu$ is an element of outwardly directed area while dv is the included volume; and the second integral being taken along a small plane closed curve of directed area $d\nu$, where rotation round $d\nu$ in the direction of the circuiting is positive. In both relations q is a quaternion function of the variable vector ρ.

In order to extend these results to integration over finite regions, we shall first suppose that the quaternion q satisfies certain conditions:—(A) that it is free from discontinuity, (B) that it is single-valued, (C) that it does not become infinite at any point of the region. Further we suppose (D) that the region included in the surface over which we propose to integrate is simply connected, so that any closed circuit drawn in that region can be made evanescent by continuous variation without cutting through the surface.

On these suppositions, we divide the region within a closed surface into infinitesimal parallelepipeds, and we apply the theorem of Art. 55 to each. Adding together the integrals $\int d\nu\,.\,q$ over the faces of these parallelepipeds, the sum obtained is equal to the sum of the corresponding elements $\nabla q\,.\,dv$; but over an interface corresponding to two parallelepipeds the directed elements are opposite, so that if one parallelepiped contributes an element $d\nu\,.\,q$, the other contributes an equal and opposite element $-d\nu\,.\,q$; consequently the sum of the integrals $\int d\nu\,.\,q$ is the integral over the bounding surface. Moreover the sum of the elements $\nabla q\,.\,dv$ is the integral $\int\nabla q\,.\,dv$ throughout the volume, and we have
$$\int d\nu\,.\,q=\int\nabla q\,.\,dv,\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{II.})$$

where the first integral is taken over the surface and the second throughout the volume.

Under the same conditions we can fill up any continuous closed curve by a net-work of parallelograms described on any surface terminated by the curve, and if these are all circuited in the same direction the elements contributed by the common sides cancel, and

$$\int d\rho \,.\, q = \int V(d\nu \,.\, \nabla) \,.\, q, \qquad\text{(III.)}$$

where $d\rho$ is a directed element of the curve and $d\nu$ a directed element of the surface. Hence it follows because (III.) has a value independent of any particular surface through the curve that over any closed surface

$$\int V(d\nu \,.\, \nabla) \,.\, q = 0. \qquad\text{(IV.)}$$

(A) Suppose a surface to exist over which q is discontinuous, and imagine the region of the volume integral to be divided into two regions by the surface of discontinuity. Applying (II.) to each of these regions and adding, we find

$$\int \nabla q dv = \int d\nu \,.\, q + \int d\nu_{12}(q_1 - q_2), \qquad\text{(V.)}$$

an element of the surface of discontinuity furnishing the parts

$$d\nu_{12} \,.\, q_1 \text{ and } d\nu_{21} \,.\, q_2, \text{ or } d\nu_{12}(q_1 - q_2).$$

(B) If q is not single-valued, it is not hard to see when infinite values of ∇q are excluded from the region that, assuming any one of its values for q at any point of the region, the value of q at every other point of the region is determinate. In fact starting from a point P with a given value of q we can return to P with a different value only if we thread some circuit along which q is indeterminate; and if q is indeterminate anywhere within the region, its corresponding deriveds must be infinite, which is contrary to supposition. When a curve locus of indeterminate values of q exists in the region, we may enclose it in a tube and so isolate it from the region. The region thus becomes multiply-connected (D).

(C) If q becomes infinite at any point, we exclude that point by a small sphere concentric with it and we take account of the surface integral over the sphere, the vectors representing the elements of directed area being drawn outwards from the region, that is, towards the centre of the sphere, and the radius of the sphere being ultimately reduced to zero.

Taking the origin at the point, the element of directed area over the surface of the sphere is $d\nu = -U\rho \,.\, r^2 \,.\, d\Omega$ if r is the radius and $d\Omega$ an element of solid angle. Then for the sphere

$$\int d\nu \,.\, q = -\int d\Omega \,.\, U\rho \,.\, r^2 \,.\, q. \qquad\text{(VI.)}$$

If over the surface of the sphere

$$q = q_0 + r^{-1} \,.\, q_1 + r^{-2} \,.\, q_2 + r^{-3} \,.\, q_3 + \text{etc.}, \qquad\text{(VII.)}$$

the surface integral vanishes unless q_2 exists, and it generally becomes infinite or indeterminate if q_3, etc., exist. Of paramount importance is the case in which q contains the term $\nabla T\rho^{-1} \,.\, e = -U\rho \,.\, T\rho^{-2} \,.\, e$. In this case if no higher negative power of r occurs, the integral becomes

$$\int d\nu \,.\, q = -\int d\Omega \,.\, e = -4\pi e, \qquad\text{(VIII.)}$$

and we must replace (II.) by

$$\int \nabla q \,.\, dv = \int d\nu \,.\, q - 4\pi e, \quad \text{(IX.)}$$

the origin being excluded from the volume integral.

In general when q_3, etc., are zero, by a well-known theorem in spherical harmonics (Art. 127) we need only consider the terms in q_2 which are linear in $U\rho$ and which we may take to be $S\alpha U\rho + \phi U\rho$. Writing $U\rho = li + mj + nk$ where l, m and n are the direction cosines of $U\rho$, and remembering that

$$\int d\Omega \,.\, l^2 = \tfrac{4}{3}\pi, \quad \int d\Omega \,.\, lm = 0, \text{ etc.},$$

we have

$$\int d\Omega \,.\, U\rho(S\alpha U\rho + \phi U\rho) = \tfrac{4}{3}\pi \Sigma i(S\alpha i + \phi i) = -\tfrac{4}{3}\pi(\alpha + m'' - 2\epsilon), \quad \text{(X.)}$$

where m'' is the first invariant and where ϵ is the spin vector of ϕ. Accordingly we must in this case replace (II.) by

$$\int \nabla q \,.\, dv = \int d\nu \,.\, q + \tfrac{4}{3}\pi(\alpha + m'' - 2\epsilon), \quad \text{(XI.)}$$

where the integral on the right is taken over the boundary and where the remaining terms are contributed by the surface of the evanescent sphere.

(D) If the region is multiply-connected we render it simply connected by drawing diaphragms* when we fall back on case (A) if q happens to be many-valued. A diaphragm corresponds to a surface of discontinuity, and $q_1 - q_2$ in (V.) becomes np where p is the cyclic increment of q and where n is an integer.

Considering now the similar cases of exception for the circuit integral, we shall suppose

(A′) that a surface of discontinuity cuts the given circuit in two points A and B. Let the surface containing the mesh-work be drawn through an arbitrary curve ACB on the surface of discontinuity. On adding the results of integration for the two circuits consisting of the part on one side of the surface of discontinuity and the curve ACB, and of the part on the other side of the surface and the curve BCA, we have exactly as in (V.)

$$\int d\rho \,.\, q + \int d\rho_{12} \,.\, (q_1 - q_2) = \int V d\nu \nabla \,.\, q. \quad \text{(XII.)}$$

It follows from (IV.) that we get exactly the same result had any other curve ADB been taken on the surface of discontinuity.

(B′) If q is not single-valued over the continuous net, its value is definite if a definite value is chosen at some one point of the net, or else q is indeterminate at some point of the net. Such a point may be surrounded by a small closed curve joined by a barrier to the circuit. The barrier must be treated as a line of discontinuity and the value of the integral round the closed curve must be taken account of.

(C′) When q becomes infinite at a point on the surface of the mesh-work, let the point be surrounded by a small circle of radius r. Then the relation becomes, when we exclude the point from the surface integral,

$$\int V d\nu \nabla \,.\, q = \int d\rho \,.\, q - \int r dU\rho \,.\, (q_0 + q_1 r^{-1} + q_2 r^{-2} + \text{etc.}), \quad \text{(XIII.)}$$

the second line integral being taken round the circle.† This integral vanishes unless there are negative powers of r. The part depending on q_1 is

$$\int dU\rho \,.\, q_1 = \int dU\rho \,.\, (S\alpha U\rho + \phi U\rho)$$

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* The interior of a hollow curtain ring becomes simply connected when a diaphragm is drawn across one normal section.

† The two line integrals are taken in the same sense of rotation round the axis of the small circle. If we choose the minus sign may be placed on the right of the sign of integration, and then we shall have the surface integral equal to the sum of two line integrals taken in opposite directions.

suppose where ϕ is a linear vector function, the terms not linear in $U\rho$ leading to a vanishing integral round the circle. Putting $U\rho = i\cos w + j\sin w$ where i and j are in the plane of the small circle, the integral easily reduces to $\pi(jSai - iSaj + j\phi i - i\phi j)$, and to $\pi(Vak - \chi' k + 2S\epsilon k)$ where χ' and ϵ have the same signification as in the chapter on linear vector functions.

Ex. 1. If $f(\nabla)$ is any linear function of the operator ∇ with constant
9 coefficients,

$$\int f(d\nu) \, . \, q = \int f(\nabla) \, . \, q \, . \, dv, \quad \int f(d\rho) \, . \, q = \int f(Vd\nu\nabla) \, . \, q,$$

and

$$\int q \, . f(d\nu) = \int q \, . f(\nabla) \, . \, dv, \quad \int q \, . f(d\rho) = \int q \, . f(Vd\nu\nabla).$$

[No step in the proof of the simpler case need be modified. In the second set of relations the operator is placed in front of the operand. See Art. 57, Ex. 11, and M'Aulay's *Utility of Quaternions in Physics.*]

Ex. 2. In general if $f(a)$ is a linear function of an arbitrary vector a while the variable vector ρ is involved in the constitution of the function, show that

$$\int f(d\nu) = \int f(\nabla) \, . \, dv, \quad \int f(d\rho) = \int f(Vd\nu\nabla),$$

where $f(\nabla)$ means that ∇ operates *in situ* on the variable vector ρ as involved in the structure of the function.

Ex. 3. Prove that $\displaystyle\int \frac{V\rho d\rho}{T\rho^3} = -\int Sd\nu\nabla \, . \, \nabla T\rho^{-1}$,

where no infinites occur.

[See Tait's *Quaternions*, Art. 504. Here the line integral is $\int Vd\rho\nabla T\rho^{-1}$, which transforms into

$$\int V \, . \, Vd\nu\nabla \, . \, \nabla T\rho^{-1} \quad \text{or} \quad \int d\nu\nabla^2 T\rho^{-1} - \int Sd\nu\nabla \, . \, \nabla T\rho^{-1}.]$$

Ex. 4. Prove that

$$\int q dv = \tfrac{1}{3}\int \rho \, . \, \nabla q \, . \, dv - \tfrac{1}{3}\int \rho d\nu q.$$

[This is an example of an extensive class of transformations depending on the invariantal properties of ∇. Transforming the surface integral, we have $\int \rho d\nu q = \int \rho(\nabla) q dv$, where ∇ operates both on ρ and on q. But $\rho\nabla = \nabla\rho = -3$. See Art. 132, p 235.]

(iii) *Inverse Operations.*

ART. **126.** We shall now establish general solutions for the equations

$$\nabla p = q, \text{ and } \nabla^2 r = q, \quad \text{.........................(I.)}$$

where q is a given quaternion function of ρ; or we shall assign definite interpretations to the functions

$$p = \nabla^{-1} q \text{ and } r = \nabla^{-2} q \quad \text{.......................(II.)}$$

for all points of an arbitrarily selected region within which infinities do not occur.

We shall first prove the transformation*

$$\int \nabla u \,.\, \nabla p \,.\, \mathrm{d}v = \int \mathrm{d}\nu \,.\, u \,.\, \nabla p - \int \nabla u_0 \nabla p \,.\, \mathrm{d}v$$

$$= \int \nabla u \,.\, \mathrm{d}\nu \,.\, p - \int \nabla (u - \mathrm{T}\rho^{-1}) \nabla \,.\, p_0 \mathrm{d}v - 4\pi p \qquad \text{(III.)}$$

in the case in which p does not become infinite within the region, while u tends to the value $\mathrm{T}\rho^{-1}$ at the origin which we suppose to be taken within the field of integration, and where $4\pi p$ in the third member is 4π times the value of p at the origin. The suffixes are intended to indicate that the affected symbols are free from the operation of ∇.

Surrounding the origin by a small sphere and supposing (∇) to operate *in situ* on u and on p we have

$$\int \nabla u \,.\, \nabla p \,.\, \mathrm{d}v = \int (\nabla) u \,.\, \nabla p \,.\, \mathrm{d}v - \int \nabla u_0 \nabla p \,.\, \mathrm{d}v$$

$$= \int \mathrm{d}\nu \,.\, u \,.\, \nabla p - \int \nabla u_0 \nabla p \,.\, \mathrm{d}v$$

for the region between the small sphere and the boundary, the surface integral over the sphere vanishing by the last Article (compare (VII.)). But these integrals may be extended throughout the entire region, for we shall show that the integrals taken through the volume of the small sphere tend to zero when the radius is indefinitely diminished. Within the sphere we may take

$$u = \mathrm{T}\rho^{-1} \text{ and } \mathrm{d}v = \mathrm{T}\rho^2 \,.\, \mathrm{d}\Omega \,.\, \mathrm{dT}\rho,$$

so that

$$\int \nabla u \,.\, \nabla p \,.\, \mathrm{d}v = -\int \mathrm{U}\rho \,.\, \nabla p \,.\, \mathrm{d}\Omega \,.\, \mathrm{dT}\rho$$

which vanishes in the limit. *A fortiori* the integral

$$\int \nabla u_0 \nabla p \,.\, \mathrm{d}v = \int \mathrm{T}\rho^{-1} \,.\, \nabla^2 p \,.\, dv$$

for the small sphere vanishes. Thus the first part of (III.) is proved.

Again for the field exclusive of the sphere

$$\int \nabla u \,.\, \nabla p \,.\, \mathrm{d}v = \int \nabla u \,.\, (\nabla) p \,.\, \mathrm{d}v - \int \nabla u \nabla \,.\, p_0 \mathrm{d}v$$

$$= \int \nabla u \,.\, \mathrm{d}\nu \,.\, p - 4\pi p - \int \nabla u \nabla \,.\, p_0 \mathrm{d}v$$

by (VIII.) of the last Article because for the surface of the sphere

$$\int \nabla u \,.\, \mathrm{d}\nu \,.\, p = +\int \mathrm{T}\rho^{-2} \,.\, \mathrm{U}\rho \,.\, \mathrm{U}\rho \,.\, \mathrm{T}\rho^2 \,.\, \mathrm{d}\Omega \,.\, p = -\int \mathrm{d}\Omega p.$$

* It is manifest from the proof of these relations that they are valid when neither p nor u become infinite in the field of integration provided we omit the term in $\mathrm{T}\rho^{-1}$ and the term $4\pi p$.

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$$\int \nabla u \nabla p \,.\, dv = \int d\nu \,.\, u \,.\, \nabla p - \int u \nabla^2 p \,.\, dv$$

$$= \int \nabla u \,.\, d\nu \,.\, p - \int \nabla^2 (u - T\rho^{-1}) \,.\, p dv - 4\pi p. \quad \text{...(III.)}'$$

Changing the origin or replacing ρ by $\rho' - \rho$ in (III.)$'$, and supposing ρ' to be the current vector in the integrations, we obtain for the particular case in which $u = T(\rho - \rho')^{-1}$ the important identities,

$$p = \int \frac{\nabla'^2 p' \,.\, dv'}{4\pi T(\rho' - \rho)} - \int \frac{d\nu' \,.\, \nabla' p'}{4\pi T(\rho' - \rho)} + \int \nabla' \,.\, \frac{1}{4\pi T(\rho' - \rho)} \,.\, d\nu' \,.\, p' \quad \text{...(IV.)}$$

$$= \nabla \int \frac{\nabla' p' dv'}{4\pi T(\rho' - \rho)} - \nabla \int \frac{d\nu' \,.\, p'}{4\pi T(\rho' - \rho)};$$

the second being deduced from (III.)$'$ by replacing ∇u by $\nabla' \,.\, T(\rho' - \rho)^{-1}$ or by its equal $-\nabla T(\rho' - \rho)^{-1}$ and taking ∇ outside the sign of integration.

If then $\nabla p = q$, we have

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$$\nabla^{-1} q = \int \frac{\nabla' q' \,.\, dv'}{4\pi T(\rho' - \rho)} - \int \frac{d\nu' \,.\, q'}{4\pi T(\rho' - \rho)} + \int \nabla' \,.\, \frac{1}{4\pi T(\rho' - \rho)} \,.\, d\nu' \,.\, p' \quad \text{(V.)}$$

$$= \nabla \int \frac{q' dv'}{4\pi T(\rho' - \rho)} - \nabla \int \frac{d\nu' \,.\, p'}{4\pi T(\rho' - \rho)};$$

and in this relation p' is any function which over the boundary satisfies $\nabla p = q$.

In like manner

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$$\nabla^{-2} q = \int \frac{q' \,.\, dv'}{4\pi T(\rho' - \rho)} - \int \frac{d\nu' \,.\, \nabla' r'}{4\pi T(\rho' - \rho)} + \int \nabla' \,.\, \frac{1}{4\pi T(\rho' - \rho)} \,.\, d\nu' \,.\, r', \quad \text{(VI.)}$$

where r is any function which over the boundary satisfies $\nabla^2 r = q$. It may be observed that in these results there is a certain analogy to the solutions of the linear function equations of Art. 65, p. 92.

If we operate on (VI.) by ∇ and put $p = \nabla r$ we find on comparison with the second form of (V.) that

$$\nabla \cdot \nabla^{-2} q = \nabla^{-1} q \quad \text{.......................(VII.)}$$

because the last integral of (VI.) vanishes under the operation of ∇ (or of $-\nabla'$ under the sign of integration operating on $\nabla' \mathrm{T}(\rho' - \rho)^{-1}$) provided ρ does not terminate on the boundary.

Ex. 1. Find the potential which produces a given distribution of force in a given field.

[If ξ is the force and P the potential, we have to determine a *scalar* function P from the equation $\xi = -\nabla P$. By (V.) this function is

$$P = -\nabla^{-1}\xi = -\int \frac{\mathrm{S}\nabla'\xi' \,.\, dv'}{4\pi \mathrm{T}(\rho - \rho')} + \int \frac{\mathrm{S}d\nu'\xi'}{4\pi \mathrm{T}(\rho - \rho')} + \int \frac{P'\mathrm{S}d\nu\nabla' \,.\, \mathrm{T}(\rho - \rho')^{-1}}{4\pi}.\Big]$$

Ex. 2. A quaternion p which satisfies the equation $\nabla^2 p = 0$ throughout a given region is expressible as a surface integral over the boundary; and a quaternion p which satisfies $\nabla p = 0$ throughout the region is of the form

$$p = -\nabla \int \frac{d\nu' \,.\, p'}{4\pi \mathrm{T}(\rho - \rho')}.$$

Ex. 3. A scalar satisfying the equation $\nabla P = 0$ is constant. A vector satisfying $\nabla\sigma = 0$ is expressible in the form $\sigma = \nabla P$ where P is a scalar function satisfying $\nabla^2 P = 0$.

Ex. 4. Construct quaternion functions of ρ, homogeneous and of the first and second orders, which shall vanish under the operation of ∇.

[For the quadratic function assume $p = \mathrm{S}\rho\phi_0\rho + \Sigma\alpha_n \mathrm{S}\rho\phi_n\rho$ where $n = 1$, 2 or 3. We have $\nabla p = -2\phi_0\rho - 2\Sigma\phi_n\rho\alpha_n$, and if $\mathrm{S}\nabla p$ is identically zero the condition $\Sigma\phi_n\alpha_n = 0$ must be satisfied. In order that $\mathrm{V}\nabla p$ may vanish, we must have $\phi_0\rho = -\Sigma\mathrm{V}\phi_n\rho\alpha_n = +\Sigma\phi_n\mathrm{V}\rho\alpha_n$ since ϕ_0 is self-conjugate. Again, because $\nabla^2 p = 0$, the first invariants of the functions ϕ must vanish. But in general $m''\mathrm{V}\rho\alpha = \mathrm{V}\rho\phi\alpha + \mathrm{V}\phi\rho\alpha + \phi\mathrm{V}\rho\alpha$, and in the present case

$$\Sigma\mathrm{V}\rho\phi_n\alpha_n + \Sigma\mathrm{V}\phi_n\rho\alpha_n + \Sigma\phi_n\mathrm{V}\rho\alpha_n = 0.$$

Hence by the former condition $-\Sigma\mathrm{V}\phi_n\rho\alpha_n$ is a self-conjugate function provided only that $\Sigma\phi_n\alpha_n = 0$, and that the first invariants are zero. Thus

$$p = -\Sigma\mathrm{S}\rho\alpha_n\phi_n\rho + \Sigma\alpha_n\mathrm{S}\rho\phi_n\rho,$$

where $m_n'' = 0$, $\Sigma\phi_n\alpha_n = 0$, vanishes under the operation of ∇.]

Ex. 5. Determine the extent of the arbitrariness in the dissection of a quaternion into the parts $\nabla^{-1}\mathrm{S}\nabla q$ and $\nabla^{-1}\mathrm{V}\nabla q$ on the supposition that $\nabla^{-1}\mathrm{S}\nabla q$ is a vector.

[The most general expressions for the parts are $\nabla^{-1}\mathrm{S}\nabla q + \sigma$ and $\nabla^{-1}\mathrm{V}\nabla q - \sigma$, where σ is a vector satisfying $\nabla\sigma = 0$. See Ex. 2.]

Ex. 6. Divide a vector σ into two parts σ_1 and σ_2 so that $\mathrm{S}\nabla\sigma_1 = 0$, $\mathrm{V}\nabla\sigma_2 = 0$.

[Here $\sigma_2 = \nabla^{-1}\mathrm{S}\nabla\sigma$ and $\sigma_1 = \nabla^{-1}\mathrm{V}\nabla\sigma$. We may calculate one of these, say σ_2 by the general formula, and the other is $\sigma - \sigma_2$.]

Ex. 7. The general solution of the equation

$$m\nabla S\nabla\sigma + n\nabla^2\sigma = \xi$$

may be written in the form

$$\sigma = \nabla^{-2}\left(\frac{\nabla^{-1}S\nabla\xi}{m+n} + \frac{\nabla^{-1}V\nabla\xi}{n}\right).$$

[The equation may be transformed into $(m+n)\nabla S\nabla\sigma + n\nabla V\nabla\sigma = \xi$, and by the last example, $\nabla S\nabla\sigma = (m+n)^{-1}\nabla^{-1}S\nabla\xi$, $\nabla V\nabla\sigma = n^{-1}\nabla^{-1}V\nabla\xi$. The solution given above of the equation of equilibrium of an elastic solid may be expressed more simply in the form $\sigma = \nabla^{-2}(n^{-1}\xi - mn^{-1}(m+n)^{-1}\nabla^{-1}S\nabla\xi)$.]

Ex. 8. If $\nabla^2 p = 0$ at all points within a closed surface, and if $\nabla^2 p_{,} = 0$ at all external points; if $p_{,} = p$ over the surface and if $p_{,}$ tends to zero at infinity,

$$p = -\int\frac{d\nu'\,.\,\nabla'(p' - p_{,}')}{4\pi T(\rho' - \rho)}.$$

[Integrating throughout external space we find if $\nabla^2 p_{,} = 0$, see note p. 219,

$$-\int d\nu'\,.\,\nabla' p_{,}'\,.\,T(\rho'-\rho)^{-1} + \int \nabla'\,.\,T(\rho'-\rho)^{-1}\,.\,d\nu'\,.\,p_{,}' = 0,$$

when ρ terminates at an internal point so that $T(\rho'-\rho)^{-1}$ does not become infinite. The surface integrals are to be taken over the closed surface and over an indefinitely large surface, but it easily appears that the latter part of the integer vanishes since $p_{,}$ vanishes at infinity. Putting $\nabla^2 p = 0$ in (IV.), remembering that $p_{,} = p$ over the closed surface, and subtracting, we have the required result.]

Ex. 9. If $f_n\rho$ is a homogeneous function of ρ of order n satisfying $\nabla^2 f_n\rho = 0$, show that

$$f_n\rho = (2n+1)\,.\int\frac{f_n\rho'\,.\,Td\nu'}{4\pi a T(\rho'-\rho)}$$

when $T\rho < a$, the integration being extended over the sphere whose centre is the origin and whose radius is a.

[The function corresponding to the $p_{,}$ of the last example is

$$f_n\rho\,.\,(aT\rho^{-1})^{2n+1}.$$

(See Art. 57, Ex. 12.) Here $\nabla'(p'-p_{,}') = (2n+1)a^{-1}U\rho'\,.\,f_n\rho'$ over the sphere and $d\nu' = U\rho' Td\nu'$.]

(iv) *Spherical Harmonics.*

ART. **127.** If $f_n(\nabla)$ is any rational and integral function of ∇, homogeneous and of order n, the function $f_n\nabla\,.\,T\rho^{-1}$ is a *solid harmonic* of order $-(n+1)$, for it is a homogeneous function of ρ which vanishes under the operation of ∇^2, the scalar operator ∇^2 being commutative in order of operation with $f_n\nabla$. Further $T\rho^{2n+1}\,.\,f_n\nabla\,.\,T\rho^{-1}$ is a solid harmonic of order n. (Art. 57, Ex. 12, p. 76.)

Because we may suppose $f_n\nabla$ to be expanded in the form

$$f_n\nabla = \Sigma a S a_1\nabla S a_2\nabla \ldots S a_n\nabla, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I.)}$$

it follows from Art. 54, Ex. 2, p. 70, that

$$T\rho^{2n+1}\,.\,f_n\nabla\,.\,T\rho^{-1} = (-)^n\,.\,1\,.\,3\,.\,5\,\ldots\,.\,(2n-1)(f_n\rho - T\rho^2\,.\,f_{n-2}\rho), \text{ (II.)}$$

where $f_{n-2}\rho$ is a determinate function of ρ, homogeneous and of order $n-2$. Hence we may expand any homogeneous function of ρ of positive order n in a series of solid harmonics, of orders n, $n-2$, $n-4$, etc.,

$$f_n\rho = \frac{\mathrm{T}\rho^{2n+1} . f_n\nabla . \mathrm{T}\rho^{-1}}{(-)^n . 1 . 3 \ldots . (2n-1)}$$

$$+\mathrm{T}\rho^2 . \frac{\mathrm{T}\rho^{2n-3} . f_{n-2}\nabla . \mathrm{T}\rho^{-1}}{(-)^{n-2} . 1 . 3 \ldots . (2n-5)} + \text{etc.}, \ldots\ldots(\text{III.})$$

where $f_{n-2}\rho$, $f_{n-4}\rho$, etc., are functions defined by equations such as (II.).

Any integral of the form $P = \int p dv . \mathrm{T}(\rho-\omega)^{-1}$ in which ω is the current vector and in which p is independent of ρ may be expressed in the form

$$P = f\nabla . \mathrm{T}\rho^{-1}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{IV.})$$

provided $\mathrm{T}\rho$ is not less than the greatest of the tensors $\mathrm{T}\omega$.

For (Art. 59 (XI.), p. 79),

$$P = \int \frac{p dv}{\mathrm{T}(\rho-\omega)} = \int p dv . e^{\mathrm{S}\omega\nabla} . \frac{1}{\mathrm{T}\rho} = f\nabla . \frac{1}{\mathrm{T}\rho}; \quad \ldots\ldots\ldots(\text{V.})$$

and we may speak of P as the potential at ρ due to a distribution of density p although it is not necessary to suppose that p is a scalar.

If $Q = \int q dv' . \mathrm{T}(\omega'-\rho)^{-1}$ is the potential of a second distribution of density q, the mutual potential is

$$W = \int \frac{pq dv dv'}{\mathrm{T}(\omega-\omega')} = \int P_{\omega'} q dv' = \int p Q_\omega dv. \ldots\ldots\ldots\ldots\ldots\ldots(\text{VI.})$$

If the second distribution lies outside a sphere of radius a having its centre at the origin and including the first distribution, we have by (V.),

$$W = \int f\nabla_{\omega'} . \frac{1}{\mathrm{T}\omega'} . q dv' = \left[\int f\nabla_{\omega'} . \frac{1}{\mathrm{T}(\rho-\omega')} . q dv'\right]_{\rho=0}$$

$$= \left[f(-\nabla) . \int \frac{q dv'}{\mathrm{T}(\rho-\omega')}\right]_{\rho=0} = f(-\nabla) . Q_0, \ \ldots\ldots\ldots(\text{VII.})$$

provided we reduce the temporary vector ρ to zero after the performance of the operations indicated, and the suffix 0 serves to remind us of this reduction.

If $Q = g_n(\rho)$ is a solid harmonic of positive order n, and if we suppose the corresponding distribution to be a surface distribution on the sphere, we may replace

$$q dv' \text{ by } (4\pi)^{-1} . (2n+1) . a^{-1} . g_n(\omega') . \mathrm{T}d\nu',$$

or by $$(4\pi)^{-1}.(2n+1).a^{n+1}.g_n(\mathrm{U}\omega).\,d\Omega,$$

utilizing Ex. 9, Art. 126, and dropping the accents as being no longer necessary. In this case (VII.) becomes

$$4\pi.f(-\nabla).g_n(\rho)_0=(2n+1)a^{n+1}\int P_\omega g_n(\mathrm{U}\omega).\,d\Omega. \quad\text{......(VIII.)}$$

In this expression it is only necessary to take account of terms of order n in $f(-\nabla)$, for $g_n(\rho)$ vanishes under the operation of terms of higher order, and the results of operation of terms of lower order vanish when ρ is reduced to zero.

If P is a solid harmonic of order $-n-1$, the form of the function $f\nabla$ is given by (III.), and

$$P=\mathrm{T}\rho^{-2n-1}.f_n\rho=\frac{f_n\nabla.\mathrm{T}\rho^{-1}}{(-)^n.1.3.\dots.(2n-1)}=f\,\nabla.\mathrm{T}\rho^{-1};\text{...(IX.)}$$

and accordingly (VIII.) becomes

$$\begin{aligned}&4\pi f_n(-\nabla).g_n\rho\\ &\quad=(-)^n.1.3.\dots.(2n-1)(2n+1).\int f_n(\mathrm{U}\omega).g_n(\mathrm{U}\omega).\,d\Omega;\ \text{...(X.)}\end{aligned}$$

while if the order of the harmonic P is $-(m+1)$ where m is not equal to n, we have

$$\int f_m(\mathrm{U}\omega).g_n(\mathrm{U}\omega).\,d\Omega=0. \quad\text{....................(XI.)}$$

Again if

$$P=\mathrm{T}(\rho-\alpha)^{-1}=e^{\mathrm{S}\alpha\nabla}.\mathrm{T}\rho^{-1}=\Sigma\mathrm{T}\alpha^n\mathrm{T}\rho^{-n-1}A_n(\mathrm{U}\rho),\text{......(XII.)}$$

we find on substitution in (VIII.),

$$\left.\begin{aligned}4\pi g_n(\mathrm{U}\alpha)&=(2n+1)\int A_n(\mathrm{U}\omega)g_n(\mathrm{U}\omega).\,d\Omega,\\ 0&=\int A_m(\mathrm{U}\omega)g_n(\mathrm{U}\omega).\,d\Omega\end{aligned}\right\}\quad\text{......(XIII.)}$$

because $$f(-\nabla).g_n(\rho)=e^{-\mathrm{S}\alpha\nabla}.g_n(\rho)=g_n(\rho+\alpha).$$

Hence we can expand any function $g(\mathrm{U}\rho)$ in a series of spherical harmonics, the harmonic of order n being

$$g_n(\mathrm{U}\alpha)=\frac{(2n+1)}{4\pi}\int A_n(\mathrm{U}\omega)g(\mathrm{U}\omega).\,d\Omega. \quad\text{..........(XIV.)}$$

Ex. 1. A scalar solid harmonic of order $-(n+1)$ may be expressed in the form $$\mathrm{S}\alpha_1\nabla.\mathrm{S}\alpha_2\nabla.\dots.\mathrm{S}\alpha_n\nabla.\mathrm{T}\rho^{-1},$$
where $\alpha_1, \alpha_2, \dots \alpha_n$ are real vectors.

[Consider the edges common to the cones $F_n\rho=0$, $\rho^2=0$. These group into conjugate pairs $\beta+\sqrt{-1}\beta'$ and $\beta-\sqrt{-1}\beta'$, and each conjugate pair lies in a real plane $\mathrm{S}\alpha\rho=0$ where $\alpha=\mathrm{V}\beta\beta'$. Having determined the vectors $\alpha_1, \alpha_2, \dots \alpha_n$ we have a relation of the form

$$F_n\rho=\rho^2F_{n-2}\rho+t.\mathrm{S}\alpha_1\rho\mathrm{S}\alpha_2\rho\dots\mathrm{S}\alpha_n\rho,$$

where t is a scalar and where $F_{n-2}\rho$ is a homogeneous function of ρ of order $n-2$. If $F_n\nabla$ is the generating operator (see (IX.)) of the harmonic we have, on putting ∇ for ρ in the above relation,

$$F_n\nabla \,.\, \mathrm{T}\rho^{-1} = t \,.\, \mathrm{S}a_1\nabla \mathrm{S}a_2\nabla \ldots \mathrm{S}a_n\nabla \,.\, \mathrm{T}\rho^{-1} \text{ because } \nabla^2\mathrm{T}\rho^{-1}=0,$$

and the scalar t can be found by comparing a coefficient.]

Ex. 2. If q is a quaternion associated with each element of mass of a body,

$$\int q\mathrm{d}m = f(\nabla) \,.\, q_0, \quad \int \tau q \mathrm{d}m = \nabla_\nabla f(\nabla) \,.\, q_0$$

where τ is the vector from a point in the body to the element $\mathrm{d}m$, where q_0 is the value of q at the origin of vectors τ, and where ∇_∇ operates on $f(\nabla)$ as if it were a function of a vector ∇.

(*a*) The first terms of the function $f(\nabla)$ are

$$f(\nabla) = M - M\mathrm{S}\tau_0\nabla + \tfrac{1}{2}\{\mathrm{S}\nabla\Phi\nabla - \tfrac{1}{2}(A+B+C)\nabla^2\} - \text{etc.},$$

where M is the mass of the body, τ_0 the vector to the centre of the mass, Φ the inertia function for the origin of vectors τ and A, B, C the principal moments of inertia for the same point.

[We have

$$\int q\mathrm{d}m = \int e^{-\mathrm{S}\tau\nabla}\mathrm{d}m \,.\, q_0 = \int (1 - \mathrm{S}\tau\nabla + \tfrac{1}{2}\mathrm{S}\tau\nabla^2 - \text{etc.})\mathrm{d}m \,.\, q_0\,;$$

and because $\quad \mathrm{S}\tau\nabla^2 = \tau^2\nabla^2 + \mathrm{V}\tau\nabla^2, \quad \int \mathrm{V}\tau\mathrm{V}\nabla\tau\mathrm{d}m = \Phi\nabla$

and $\quad \int \tau^2\mathrm{d}m = -\tfrac{1}{2}(A+B+C),$

the expansion is justified. Again the differential of fa corresponding to $\mathrm{d}a$ is

$$\mathrm{d}fa = -\mathrm{S}\mathrm{d}a\nabla_a \,.\, fa = -\int \mathrm{S}\mathrm{d}a\tau e^{-\mathrm{S}\tau a}\mathrm{d}m.]$$

Ex. 3. A heavy body is placed in a field in which the gravitational potential is P. The potential energy of the body (W), the resultant force and the resultant couple (λ and μ) acting on the body and referred to its centre of mass, are

$$W = MP + \tfrac{1}{2}\mathrm{S}\nabla\Phi\nabla \,.\, P, \quad \lambda = M\nabla P + \tfrac{1}{2}\mathrm{S}\nabla\Phi\nabla \,.\, \nabla P, \quad \mu = \mathrm{V}\Phi\nabla\nabla \,.\, P.$$

(v) *Various expressions for* ∇.

ART. **128.** We shall now examine in greater detail than in Art. 57 the various analytical expressions for the operator ∇ and for ∇^2.

In terms of three arbitrary differentials we may write

$$\nabla = \lambda\mathrm{d} + \lambda'\mathrm{d}' + \lambda''\mathrm{d}'', \quad \text{(I.)}$$

where (Art. 54 (VI.), p. 70)

$$\lambda = -\frac{\mathrm{Vd}'\rho\mathrm{d}''\rho}{\mathrm{Sd}\rho\mathrm{d}'\rho\mathrm{d}''\rho}, \quad \lambda' = -\frac{\mathrm{Vd}''\rho\mathrm{d}\rho}{\mathrm{Sd}\rho\mathrm{d}'\rho\mathrm{d}''\rho}, \quad \lambda'' = -\frac{\mathrm{Vd}\rho\mathrm{d}'\rho}{\mathrm{Sd}\rho\mathrm{d}'\rho\mathrm{d}''\rho}. \quad \text{(II.)}$$

The operator ∇^2 is now

$$\nabla^2 = \Sigma\lambda^2\mathrm{d}^2 + \Sigma(\lambda'\lambda''\mathrm{d}'\mathrm{d}'' + \lambda''\lambda'\mathrm{d}''\mathrm{d}') + \Sigma\nabla\lambda \,.\, \mathrm{d}, \quad \text{(III.)}$$

and in the third sum ∇ operates on the vectors λ alone and not on the operand of ∇^2.

Remembering that ∇^2 is a scalar operator, this breaks up into the two parts

$$\nabla^2 = \Sigma\lambda^2 d^2 + \Sigma S\lambda'\lambda''(d'd'' + d''d') + \Sigma S\nabla\lambda \,.\, d \,; \qquad \text{(IV.)}$$

$$0 = \Sigma V\lambda'\lambda'' \,.\, (d'd'' - d''d') + \Sigma V\nabla\lambda \,.\, d. \qquad \text{(V.)}$$

It is only when the differentials are *independent* that the order in which the differentiations are performed is indifferent, and it is only in this case that we can generally suppress the terms involving $d'd'' - d''d'$ and similar expressions in (V.).

When independent differentials are employed, we use the expression (Art. 57. (III.), p. 74),

$$\nabla = -\frac{V\rho_2\rho_3}{S\rho_1\rho_2\rho_3} \cdot \frac{\partial}{\partial u} - \frac{V\rho_3\rho_1}{S\rho_1\rho_2\rho_3} \cdot \frac{\partial}{\partial v} - \frac{V\rho_1\rho_2}{S\rho_1\rho_2\rho_3} \cdot \frac{\partial}{\partial w} \,; \qquad \text{(VI.)}$$

or as it may be briefly written

$$\nabla = \nu_1\frac{\partial}{\partial u} + \nu_2\frac{\partial}{\partial v} + \nu_3\frac{\partial}{\partial w}, \qquad \text{(VII.)}$$

where the vectors ν_1, ν_2 and ν_3 satisfy the relations

$$S\nu_1\rho_1 + 1 = 0, \text{ etc.}, \quad S\nu_2\rho_3 = 0, \quad S\nu_3\rho_2 = 0, \text{ etc.}; \qquad \text{(VIII.)}$$

or again we may put

$$\nabla = \nabla u \,.\, \frac{\partial}{\partial u} + \nabla v \,.\, \frac{\partial}{\partial v} + \nabla w \,.\, \frac{\partial}{\partial w}, \qquad \text{(IX.)}$$

as we see by comparing the results of operation of the forms (VII.) and (IX.) on u, v and w. Thus

$$\nu_1 = \nabla u, \;\; \nu_2 = \nabla v, \;\; \nu_3 = \nabla w \qquad \text{(X.)}$$

and

$$V\nabla\nu_1 = 0, \;\; V\nabla\nu_2 = 0, \;\; V\nabla\nu_3 = 0. \qquad \text{(XI.)}$$

The vectors ν_1, ν_2 and ν_3 are the normals at the extremity of ρ to the three surfaces $u = const.$, $v = const.$ and $w = const.$ which pass through that point.

The appropriate expressions for ∇^2 are now

$$\nabla^2 = \Sigma\nu_1^2\frac{\partial^2}{\partial u^2} + 2\Sigma S\nu_2\nu_3 \,.\, \frac{\partial^2}{\partial v\partial w} + \Sigma S\nabla\nu_1 \,.\, \frac{\partial}{\partial u} \,; \qquad \text{(XII.)}$$

or

$$\nabla^2 = \Sigma(\nabla u)^2 \,.\, \frac{\partial^2}{\partial u^2} + 2S\nabla v\nabla w \,.\, \frac{\partial^2}{\partial v\partial w} + \Sigma\nabla^2 u \,.\, \frac{\partial}{\partial u}. \qquad \text{(XIII.)}$$

Again introducing the operand q for the sake of greater clearness, we may write

$$\nabla q = -\frac{\left\{\frac{\partial}{\partial u}(V\rho_2\rho_3 \cdot q) + \frac{\partial}{\partial v}(V\rho_3\rho_1 \cdot q) + \frac{\partial}{\partial w}(V\rho_1\rho_2 \cdot q)\right\}}{S\rho_1\rho_2\rho_3}, \qquad \text{(XIV.)}$$

because the terms which involve the second deriveds of ρ, such as

$V\rho_{12}\rho_3 \cdot q + V\rho_3\rho_{12} \cdot q$, cancel in pairs. Operating with this form of ∇ on (VI.), we have

$$\nabla^2 = +\frac{1}{S\rho_1\rho_2\rho_3}\left\{\Sigma\frac{\partial}{\partial u}\left(\frac{V\rho_2\rho_3{}^2}{S\rho_1\rho_2\rho_3}\cdot\frac{\partial}{\partial u}\right)+\Sigma\frac{\partial}{\partial u}\left(\frac{V\rho_2\rho_3 \cdot V\rho_3\rho_1}{S\rho_1\rho_2\rho_3}\cdot\frac{\partial}{\partial v}\right)\right\}, \text{(XV.)}$$

where the second sum contains six terms, and to this the sign S may be prefixed. Or in terms of the vectors ν it easily appears that this reduces to

$$\nabla^2 = +S\nu_1\nu_2\nu_3\left\{\Sigma\frac{\partial}{\partial u}\left(\frac{\nu_1{}^2}{S\nu_1\nu_2\nu_3}\cdot\frac{\partial}{\partial u}\right)+\Sigma\frac{\partial}{\partial u}\left(\frac{S\nu_1\nu_2}{S\nu_1\nu_2\nu_3}\cdot\frac{\partial}{\partial v}\right)\right\}. \quad \text{(XVI.)}$$

Whenever the surfaces, $u = const.$, $v = const.$ and $w = const.$, are equipotential surfaces with the corresponding potentials, u, v and w, the operator ∇^2 is a homogeneous quadratic in the differentiating symbols $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}$. This property follows directly from (XIII.). The converse is also true.

When the surfaces are mutually rectangular, the operator ∇^2 is independent of the products of differentiating symbols. In this case we find from (XV.) the most convenient expression for ∇^2 to be

$$\nabla^2 = -\frac{1}{T \cdot \rho_1\rho_2\rho_3}\cdot\Sigma\frac{\partial}{\partial u}\left(T\cdot\frac{\rho_2\rho_3}{\rho_1}\cdot\frac{\partial}{\partial u}\right). \quad \text{.........(XVII.)}$$

Ex. 1. Determine expressions for ∇^2 where

(1) $\rho = u\{(i\cos w + j\sin w)\sin v + k\cos v\}$;

(2) $\rho = u(i\cos v + j\sin v) + kw$;

(3) $\rho = \sqrt{\{(\phi+u)(\phi+v)(\phi+w)\}}\cdot\epsilon$, as in Art. 84.

Ex. 2. If a scalar function P of a scalar function u of ρ can be found to satisfy $\nabla^2 P = 0$, show that

$$(\nabla u)^2\cdot\frac{\partial^2 P}{\partial u^2}+\nabla^2 u\cdot\frac{\partial P}{\partial u}=0 \quad\text{and}\quad V\nabla u\nabla\cdot\frac{\nabla^2 u}{(\nabla u)^2}=0.$$

[See (XIII.) for the first condition. The second expresses that $\nabla^2 u\,.\,(\nabla u)^{-2}$ is a function of u.]

Ex. 3. Given that a family of surfaces $u = \text{const.}$ is an equipotential system, show that the potential corresponding to u is

$$P = \int du \,.\, e^{-\int\frac{\nabla^2 u}{(\nabla u)^2}\cdot du}.$$

[See the last example.]

Ex. 4. A family of concentric, similar and coaxial quadrics compose an equipotential system. Show that the sum of the reciprocals of the squares of their principal axes is zero, or else the quadrics are spheres. Determine also the corresponding potentials.

[Here $u = \frac{1}{2}S\rho\phi\rho$, $\nabla u = -\phi\rho$, $\nabla^2 u = m''$. The condition of Ex. 2 becomes $V\phi\rho\phi^2\rho\,.\,m'' = 0$.]

Ex. 5. Find the condition that the family of surfaces $f(\rho, u)=0$ should form an equipotential system, and determine the potential when the condition is satisfied.

[Imagine u to be expressed as a function of ρ by solution of the equation $f(\rho, u)=0$. On this understanding we may treat $f(\rho, u)=0$ as an identity and equate to zero the results of operating on it by ∇ and ∇^2. We find

$$\nabla f+\nabla u\,.\,\frac{\partial f}{\partial u}=0, \quad \nabla^2 f+2S\nabla u\frac{\partial \nabla f}{\partial u}+\nabla^2 u\frac{\partial f}{\partial u}+(\nabla u)^2\,.\,\frac{\partial^2 f}{\partial u^2}=0,$$

where ∇ operates on f as if f were a function of ρ alone, and where consequently ∇ and $\frac{\partial}{\partial u}$ are commutative in order of operation on f. Utilizing the results of Ex. 2 to eliminate $\nabla^2 u$ and eliminating ∇u we find

$$\frac{\partial}{\partial u}\log\frac{\partial P}{\partial u}=\frac{\partial}{\partial u}\cdot\log\left(\frac{\partial f}{\partial u}\cdot\frac{1}{(\nabla f)^2}\right)+\frac{\nabla^2 f}{(\nabla f)^2}\cdot\frac{\partial f}{\partial u}.$$

The condition to be satisfied is that the right-hand member—a function of ρ and u—should reduce to a function of u alone by aid of the equation $f(\rho, u)=0$. If $F(\rho, u)$ reduces to a function of u alone by aid of the equation $f(\rho, u)=0$, we must have $\nabla F+\nabla u\,.\,\frac{\partial F}{\partial u} \parallel \nabla u \parallel \nabla f$ or simply $V\nabla f\nabla F=0$. Thus the condition required is

$$V\nabla f\nabla\left\{\frac{\partial}{\partial u}\cdot\log\left(\frac{\partial f}{\partial u}\cdot\frac{1}{(\nabla f)^2}\right)+\frac{\nabla^2 f}{(\nabla f)^2}\cdot\frac{\partial f}{\partial u}\cdot\right\}=0.]$$

Ex. 6. Show that the family of confocals $S\rho(\phi+u)^{-1}\rho+1=0$ is an equipotential system, and determine the potential.

[Here we have $\quad \nabla f=-2(\phi+u)^{-1}\rho$ and $\frac{\partial f}{\partial u}=-(\phi+u)^{-1}\rho^2=-\frac{1}{4}(\nabla f)^2$;

also $$\nabla^2 f=-2\Sigma i(\phi+u)^{-1}i=2\Sigma(a^2+u)^{-1}.$$

These give

$$\frac{\partial}{\partial u}\log\frac{\partial P}{\partial u}=-\tfrac{1}{2}\Sigma\frac{1}{a^2+u}=-\frac{\partial}{\partial u}\log\sqrt{\{(a^2+u)(b^2+u)(c^2+u)\}},$$

and $$P=P_0\int\frac{du}{\sqrt{\{(a^2+u)(b^2+u)(c^2+u)\}}}.]$$

Ex. 7. The condition that the family of surfaces $f(\rho, u)=0$ should compose a system of characteristic surfaces in an optical medium of constant density is

$$V\nabla f\nabla\left\{(\nabla f)^2\left(\frac{\partial f}{\partial u}\right)^{-2}\right\}=0.$$

[Hamilton's characteristic functions satisfy the relation $T\nabla Q=n$, where n is the index of refraction of the medium. If the family of surfaces satisfies the condition we must have Q a function of u, so that $\nabla Q=Q'\nabla u=-Q'f'^{-1}\nabla f$, where the accents denote differentiation with respect to u. Hence when n is constant, $T\nabla f\,.\,f'^{-1}$ must reduce to a function of u, or $V\nabla f\nabla(T\nabla f\,.\,f'^{-1})=0.]$

(vi) *Kinematics of a deformable system.*

ART. **129.** If q is any function of ρ and t, its total differential may be written in the form

$$dq=\dot{q}dt-Sd\rho\nabla\,.\,q\,; \quad\text{.......................(I.)}$$

and in particular when we replace $\mathrm{d}\rho$ by $\sigma \mathrm{d}t$ we shall write

$$\mathrm{D}q = \dot{q}\mathrm{d}t - \mathrm{S}\sigma\nabla \,.\, q \,.\, \mathrm{d}t \text{ and } \mathrm{D}\,q = \dot{q} - \mathrm{S}\sigma\nabla \,.\, q. \quad \text{.........(II.)}$$

When σ denotes a velocity, $\mathrm{D}_t q$ is the rate of change of the quantity q regarded as associated with the moving point. On the other hand $\dot{q}$ is the rate in change of q at a fixed point, and $-\mathrm{S}\mathrm{d}\rho\nabla \,.\, q$ is the change in the value q from the extremity of ρ to that of $\rho + \mathrm{d}\rho$ at a given instant.

If $\mathrm{d}\rho$, $\mathrm{d}\nu$ and $\mathrm{d}v$ are elements of directed line, directed area and volume respectively, at the extremity of ρ in a medium moving with the velocity σ, we have by Art. 124 (II.), p. 212,

$$\left.\begin{aligned} \mathrm{D}_t(q\mathrm{d}v) &= (\mathrm{D}_t q + m''q) \,.\, \mathrm{d}v, \\ \mathrm{D}_t(\mathrm{S}\varpi\mathrm{d}\nu) &= \mathrm{S}(\mathrm{D}_t\varpi + \chi\varpi) \,.\, \mathrm{d}\nu = \mathrm{S}\dot{\underline{\varpi}}\mathrm{d}\nu, \\ \mathrm{D}_t(\mathrm{S}\varpi\mathrm{d}\rho) &= \mathrm{S}(\mathrm{D}_t\varpi + \phi'\varpi) \,.\, \mathrm{d}\rho = \mathrm{S}\dot{\underline{\underline{\varpi}}}\mathrm{d}\rho, \end{aligned}\right\} \quad \text{.........(III.)}$$

where* (Art. 124 (I.) and (III.))

$$\left.\begin{aligned} \dot{\underline{\varpi}} &= \mathrm{D}_t\varpi + \chi\varpi = \mathrm{D}_t\varpi - \mathrm{V}\nabla\mathrm{V}\sigma\varpi_0, \\ \dot{\underline{\underline{\varpi}}} &= \mathrm{D}_t\varpi + \phi'\varpi = \mathrm{D}_t\varpi - \nabla\mathrm{S}\sigma\varpi_0, \end{aligned}\right\} \quad \text{.............(IV.)}$$

because for example we have $\mathrm{S}\varpi\mathrm{D}_t\mathrm{d}\nu = \mathrm{S}\varpi\chi'\mathrm{d}\nu = \mathrm{S}\chi\varpi\mathrm{d}\nu$.

In terms of the spin-vector $\epsilon = \frac{1}{2}\mathrm{V}\nabla\sigma$, the divergence $m'' = -\mathrm{S}\nabla\sigma$ and the self-conjugate part ϕ_0 of ϕ we may also write

$$\dot{\underline{\varpi}} = \mathrm{D}_t\varpi - \mathrm{V}\epsilon\varpi + (m'' - \phi_0)\varpi, \quad \dot{\underline{\underline{\varpi}}} = \mathrm{D}_t\varpi - \mathrm{V}\epsilon\varpi + \phi_0\varpi\,; \quad \text{....(V.)}$$

or explicitly in terms of σ we have

$$\dot{\underline{\varpi}} = \dot{\varpi} - \mathrm{V}\nabla\mathrm{V}\sigma\varpi - \sigma\mathrm{S}\nabla\varpi, \quad \dot{\underline{\underline{\varpi}}} = \dot{\varpi} - \nabla\mathrm{S}\sigma\varpi - \mathrm{V}\sigma\mathrm{V}\nabla\varpi. \quad \text{....(VI.)}$$

To prove these results observe that

$$\begin{aligned} \dot{\underline{\varpi}} &= \dot{\varpi} - \mathrm{S}\sigma_0\nabla \,.\, \varpi - \varpi_0\mathrm{S}\nabla\sigma + \mathrm{S}\varpi_0\nabla \,.\, \sigma \\ &= \dot{\varpi} - \mathrm{S}\sigma(\nabla) \,.\, \varpi + \mathrm{S}\varpi(\nabla) \,.\, \sigma - \sigma_0\mathrm{S}\nabla\varpi \end{aligned}$$

and that

$$\dot{\underline{\underline{\varpi}}} = \dot{\varpi} - \mathrm{S}\sigma_0\nabla \,.\, \varpi - \nabla\mathrm{S}\varpi_0\sigma = \dot{\varpi} - \mathrm{S}\sigma_0\nabla \,.\, \varpi + \nabla\mathrm{S}\varpi\sigma_0 - \nabla\mathrm{S}\sigma\varpi,$$

where (∇) operates *in situ* both on σ and ϖ and where σ_0 and ϖ_0 are free from the operation of ∇.

In addition we may write

$$(\mathrm{D}_t + m'')q = \dot{q} - \mathrm{S}\sigma(\nabla) \,.\, q \quad \text{...................(VII.)}$$

because this expression is $\dot{q} - \mathrm{S}\sigma_0\nabla \,.\, q - \mathrm{S}\nabla\sigma \,.\, q_0$.

We may connect this with previous results by observing that

$$(\mathrm{D}_t + m'')\mathrm{S}\varpi\omega = \mathrm{S}(\dot{\underline{\varpi}}\omega + \varpi\dot{\underline{\underline{\omega}}}) = \mathrm{S}(\dot{\underline{\underline{\varpi}}}\omega + \varpi\dot{\underline{\omega}}) \quad \text{.......(VIII.)}$$

is a consequence of (IV.) where ω is any vector function of ρ and t.

Also

$$\mathrm{S}\nabla\dot{\underline{\varpi}} = (\mathrm{D}_t + m'')\mathrm{S}\nabla\varpi. \quad \text{....................(IX.)}$$

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* See H. A. Lorentz, *Encyklopädie der math. Wiss.*, V_2, p. 75.

We may also observe that if $\omega = V\nabla\varpi$, we have by (VI.)

$$V\nabla\dot{\underline{\underline{\varpi}}} = V\nabla\dot{\varpi} - V\nabla V\sigma V\nabla\varpi = \dot{\omega} - V\nabla V\sigma\omega, \quad S\nabla\omega = 0;$$

since t and ρ are independent, so that the order of operation by ∇ and of partial differentiation with respect to t is indifferent.

Hence $$V\nabla\dot{\underline{\underline{\varpi}}} = \dot{\underline{\omega}}, \text{ if } \omega = V\nabla\varpi. \quad \text{.......................(X.)}$$

From these relations we derive various forms for *equations of continuity*; and the voluminal, the areal and the linear equations of continuity are respectively

$$(D_t + m'')q = 0, \quad \dot{\underline{\varpi}} = 0, \quad \dot{\underline{\underline{\varpi}}} = 0. \quad \text{..................(XI.)}$$

The first asserts that $q\mathrm{d}v$ does not change for the element of volume; the second requires $S\varpi\mathrm{d}\nu$ to remain constant for all vector areas $\mathrm{d}\nu$, and $S\varpi\mathrm{d}\rho$ remains unchanged if $\dot{\underline{\underline{\varpi}}} = 0$.

Instead of supposing the quantities q, ϖ and σ to be functions of ρ and t, we may take them to be functions of t, u, v and w where u, v and w are three parameters which individualize the moving point.

This is Lagrange's method, and Euler's method is that in which everything is expressed in terms of ρ and t. The total differential of q we shall now write in the form,

$$Dq = \frac{\partial q}{\partial t}\mathrm{d}t + \frac{\partial q}{\partial u}\mathrm{d}u + \frac{\partial q}{\partial v}\mathrm{d}v + \frac{\partial q}{\partial w}\mathrm{d}w; \quad \text{................(XII.)}$$

and following the moving point we have

$$D_t q = \frac{\partial q}{\partial t} \quad \text{...........................(XIII.)}$$

since u, v and w remain unchanged. In particular

$$\sigma = \frac{\partial\rho}{\partial t}, \quad D_t\sigma = \frac{\partial^2\rho}{\partial t^2}.$$

The vectors $\dot{\underline{\varpi}}$ and $\dot{\underline{\underline{\varpi}}}$ now become

$$\dot{\underline{\varpi}} = \frac{\partial\varpi}{\partial t} - V\nabla V\frac{\partial\rho}{\partial t}\varpi_0, \quad \dot{\underline{\underline{\varpi}}} = \frac{\partial\varpi}{\partial t} - \nabla S\frac{\partial\rho}{\partial t}\varpi_0, \quad \text{......(XIV.)}$$

as appears on reference to (IV.). The appropriate form for ∇ in these relations is that given in Art. 128 (VI.) or (XIV.). The element of volume is now $-S\rho_1\rho_2\rho_3 \mathrm{d}u\mathrm{d}v\mathrm{d}w$, and the voluminal equation of continuity is simply (compare (III.))

$$qS\rho_1\rho_2\rho_3 = const. \quad \text{.......................(XV.)}$$

Ex. 1. If c is the density of a continuous distribution of matter moving with the velocity σ, Euler's equation of continuity is

$$\dot{c} = S\nabla(c\sigma) \quad \text{or} \quad D_t c = cS\nabla\sigma;$$

and Lagrange's equation is

$$-c\mathrm{S}\rho_1\rho_2\rho_3 = C = const.$$

(*a*) Hence $\qquad \mathrm{D}_t \log c = -\dfrac{\partial}{\partial t}\log \mathrm{S}\rho_1\rho_2\rho_3 = \mathrm{S}\nabla\dfrac{\partial\rho}{\partial t} = \mathrm{S}\nabla\sigma.$

Ex. 2. Show that

$$\underline{\dot{\sigma}} = \dot{\sigma} - \sigma\mathrm{S}\nabla\sigma, \quad \underline{\underline{\dot{\sigma}}} = \dot{\sigma} - \nabla\sigma^2 - \mathrm{V}\sigma\mathrm{V}\nabla\,\sigma = \mathrm{D}_t\sigma - \tfrac{1}{2}\nabla\,.\,\sigma^2.$$

Ex. 3. Show that $\qquad \dfrac{\partial\nabla}{\partial t}\,.\,\rho = -\nabla\dfrac{\partial\rho}{\partial t}.$

Ex. 4. In general

$$\mathrm{V}\underline{\underline{\dot{\varpi}}}\varpi' + \mathrm{V}\varpi\underline{\underline{\dot{\varpi}}}' = \underline{\dot{\varpi}}'', \quad \mathrm{V}\underline{\dot{\varpi}}\varpi' + \mathrm{V}\varpi\underline{\dot{\varpi}}' = m''\varpi'' + \underline{\underline{\dot{\varpi}}}'' \quad \text{where} \quad \varpi'' = \mathrm{V}\varpi\varpi'.$$

[These relations follow most easily from (IV.).]

ART. **130**. The integral

$$F = -\int \mathrm{S}\varpi\mathrm{d}\rho \qquad \text{..........................(I.)}$$

taken from one point to another along a curve depends generally on the nature of the curve; but if $\mathrm{V}\nabla\varpi = 0$, so that $\varpi = \nabla P$, the value of the integral is simply the difference of the values of P at the extremities of the curve. This integral may be called the *flow* of the vector ϖ along the curve.

The time rate of change of F as the curve moves with the medium with velocity σ is

$$\mathrm{D}_t F = -\int \mathrm{S}\underline{\underline{\dot{\varpi}}}\mathrm{d}\rho, \qquad \text{..........................(II.)}$$

and if this integral is independent of the nature of the curve,

$$\underline{\underline{\dot{\varpi}}} = \nabla Q, \;\; \dot{\varpi} - \mathrm{V}\sigma\mathrm{V}\nabla\varpi = \nabla(\mathrm{S}\sigma\varpi + Q), \;\; \mathrm{D}_t\varpi = \nabla(\mathrm{S}\sigma\varpi_0 + Q) \quad \text{(III.)}$$

are different forms of the condition to be satisfied, Q being a scalar function of ρ and t. Other forms of the condition are

$$\mathrm{V}\nabla\underline{\underline{\dot{\varpi}}} = 0, \;\; \mathrm{V}\nabla\dot{\varpi} - \mathrm{V}\nabla\mathrm{V}\sigma\mathrm{V}\nabla\varpi = 0, \;\; \mathrm{V}\nabla\mathrm{D}_t\varpi = \mathrm{V}\nabla'\nabla\mathrm{S}\sigma\varpi'; \quad \text{(IV.)}$$

or again (Art. 129 (X.))

$$\underline{\dot{\omega}} = 0, \;\; \text{where} \;\; \omega = \mathrm{V}\nabla\varpi. \qquad \text{....................(V.)}$$

As regards the third of (IV.), note that $\mathrm{V}\nabla^2\mathrm{S}\sigma\varpi_0 = 0$.

In general we have (Art. 129 (VI.))

$$\mathrm{D}_t F = -\int \mathrm{S}(\dot{\varpi} - \mathrm{V}\sigma\mathrm{V}\nabla\varpi)\mathrm{d}\rho - [\mathrm{S}\sigma\varpi], \qquad \text{...........(VI.)}$$

and

$$\mathrm{D}F = -\int \mathrm{S}\dot{\varpi}\mathrm{d}\rho\mathrm{d}t - \int \mathrm{S}\mathrm{V}\nabla\varpi\mathrm{d}\nu - [\mathrm{S}\sigma\varpi]\mathrm{d}t, \;\; \text{where } \mathrm{d}\nu = \mathrm{V}\sigma\mathrm{d}\rho\mathrm{d}t \quad \text{(VII.)}$$

and where $[\mathrm{S}\sigma\varpi]$ denotes the difference of the values of $\mathrm{S}\sigma\varpi$ at the extremities of the curve. The expression for $\mathrm{D}F$ shows the meaning of the various terms, $\mathrm{d}\nu$ being an element of the area swept out by an element of the curve in the time $\mathrm{d}t$.

In the case of a closed curve, the *circulation* of the vector ϖ in the curve and its rate of change are expressed by

$$C=-\int S\varpi d\rho, \quad D_t C=-\int S\underline{\dot{\varpi}} d\rho; \quad \text{............(VIII.)}$$

or when ϖ does not become infinite at any point of a surface drawn over the circuit, we may transform the circulation into a surface integral so that

$$C=-\int S\omega d\nu, \quad D_t C=-\int S\underline{\dot{\omega}} d\nu, \quad \omega=V\nabla\varpi. \quad \text{.........(IX.)}$$

The circulation is therefore the *flux* of the vector $\omega(=V\nabla\varpi)$ through the circuit, and the rate of change of the circulation is the flux of the derived vector $\underline{\dot{\omega}}$ $(=V\nabla\underline{\dot{\varpi}})$ or the circulation of $\underline{\dot{\varpi}}$.

For any small plane circuit, the circulation $-SV\nabla\varpi d\nu$ is the projection of $V\nabla\varpi$ on the normal to the circuit into the area of the circuit. Thus $V\nabla\varpi$ determines the aspect of the unit circuit in which the circulation is a maximum, and it likewise gives the magnitude of the circulation $TV\nabla\varpi$ in that *principal circuit.* In like manner $\underline{\omega}$ determines the aspect of the circuit in which the rate of change of circulation is a maximum as well as the value of that maximum.

The vector $D_t V\nabla\varpi$ determines the rate of change of the circulation from one principal circuit to another following the motion of the medium. A principal circuit does not generally remain a principal circuit. We note that by (IV.) and by Art. 129 (IV.)

$$\underline{\dot{\omega}}=V\nabla D_t\varpi-V\nabla'\nabla S\sigma\varpi'=D_t V\nabla\varpi-V\nabla V\sigma V\nabla'\varpi'; \quad \text{......(X.)}$$

and in general we have

$$(D_t\nabla-\nabla D_t)\,.\,q=\nabla' S\sigma'\nabla\,.\,q, \quad \text{...........(XI.)}$$

because $D_t\nabla\,.\,q=\nabla\dot{q}-S\sigma\nabla\,.\,\nabla q$, $\nabla D_t q=\nabla\dot{q}-(\nabla)S\sigma\nabla\,.\,q$.

If a tubular surface, drawn through any circuit, is composed of curves satisfying the differential equation

$$V\nabla\varpi d\rho=0; \quad \text{..........................(XII.)}$$

or, what is equivalent, if

$$S\nabla\varpi d\nu=0 \quad \text{..........................(XIII.)}$$

over the tubular surface, the circulation in any evanescible* circuit traced on this surface is zero. In particular if ABC and A′B′C′ are two circuits embracing the tube, the circuit ABCAA′B′C′A′A is evanescible and also the circuit AA′A. From this it follows that the circulation in ABCA is equal to that in A′B′C′A′, being opposite to that in A′C′B′A′. Hence the circulation

* An evanescible circuit may be reduced to zero by continuous variation.

is the same in all circuits drawn on the tube so as to embrace it once.

The *flux* of the vector ϖ through a given surface bounded by a given curve is

$$G=-\int S\varpi d\nu, \quad\text{.......................(XIV.)}$$

and the condition that this should depend only on the bounding curve is that the divergence of ϖ should vanish, or

$$S\nabla\varpi=0, \quad\text{..........................(XV.)}$$

as we see by transforming the integral over a closed surface into a volume integral.

The rate of change of the flux is

$$D_tG=-\int S\underline{\dot{\varpi}}d\nu, \quad\text{.......................(XVI.)}$$

and the condition that this rate of change should depend only on the bounding curve is

$$S\nabla\underline{\dot{\varpi}}=0 \text{ or } S\nabla\dot{\varpi}-S(\nabla)\sigma\,.\,S\nabla\varpi=0, \text{ or } (D_t+m'')S\nabla\varpi=0. \text{ (XVII.)}$$

In any case in which $S\nabla\varpi=0$, if a tube is constructed of the lines $Vd\rho\varpi=0$ through a circuit, the fluxes across all sections of the tube are the same, and the value of the flux is the *strength* of the tube. For a small tube we have, if $Td\nu$ is the area of a cross section and if dn is the strength,

$$Td\nu T\varpi=dn, \quad \text{where } S\nabla\varpi=0. \quad\text{...........(XVIII.)}$$

Ex. 1. If $V\nabla D_t\sigma=0$, the circulation of the vectors σ in any circuit moving with the medium remains unchanged.

[See (III.) and (IV.). We have $D_t\sigma=\nabla(\frac{1}{2}\sigma^2+Q)$.]

Ex. 2. Show that in Lagrange's method

$$(D_t\nabla-\nabla D_t)q=\frac{\partial\nabla}{\partial t}\cdot q.$$

ART. **131.** In Art. 126 we showed that any vector ϖ can be expressed in the form (see (IV.), p. 220)

$$\varpi=\nabla p, \quad (S\nabla p=0), \quad\text{..........................(I.)}$$

where p is a certain quaternion. We shall examine how this quaternion is related to the flow and the flux of the vector ϖ. In terms of p,

$$F=-\int S\varpi d\rho=-\int SV\nabla Vp\,.\,d\rho+[Sp], \quad\text{...........(II.)}$$

because $-S\,.\,\nabla Sp\,.\,d\rho=dSp$. Hence for a closed circuit, the circulation depends merely on Vp. If the circulation in every circuit vanishes, the quaternion p reduces to a scalar, as we have already observed. The circulation in general is expressible as

$$C=-\int S\,.\,V\nabla Vp\,.\,d\rho=-\int S\,.\,\nabla^2 Vp\,.\,d\nu. \quad\text{..........(III.)}$$

We have also

$$\mathrm{D}_t F = -\int \mathrm{S}.(\nabla \mathrm{V}\dot{p} - \nabla \mathrm{S}\sigma\nabla\mathrm{V}p - \sigma\nabla^2\mathrm{V}p)\mathrm{d}\rho + [\mathrm{D}_t \mathrm{S}p]; \quad \text{.....(IV.)}$$

and $\dot{\underline{\underline{\varpi}}}$ and $\dot{\underline{\varpi}}$ are

$$\dot{\underline{\underline{\varpi}}} = \nabla\dot{p} - \nabla\mathrm{S}\sigma\nabla p - \mathrm{V}\sigma\nabla^2\mathrm{V}p, \quad \dot{\underline{\varpi}} = \nabla\dot{p} - \mathrm{V}\nabla\mathrm{V}\sigma\nabla p - \sigma\nabla^2\mathrm{S}p. \quad \text{(V.)}$$

The flux is

$$G = -\int \mathrm{S}\varpi\mathrm{d}\nu = -\int \mathrm{Sd}\nu\nabla.\mathrm{S}p - \int \mathrm{SV}p\mathrm{d}\rho, \quad \text{..........(VI.)}$$

since

$$\int \mathrm{Sd}\nu\nabla\mathrm{V}p = \int \mathrm{Sd}\rho\mathrm{V}p.$$

The flux through any closed surface depends merely on $\mathrm{S}p$. Comparing (II.) and (VI.) we see that $\mathrm{V}p$ and $\mathrm{S}p$ play a complementary rôle in these two relations. Various forms may be found for $\mathrm{D}_t G$ on which we cannot delay.

Replacing p by ϖ in the second form of the identity (Art. 126 (IV.)), we obtain the expression

$$\varpi = \nabla\int\frac{\nabla'\varpi'\mathrm{d}v'}{4\pi\mathrm{T}(\rho'-\rho)} - \nabla\int\frac{\mathrm{d}\nu'\varpi'}{4\pi\mathrm{T}(\rho'-\rho)} \quad \text{..............(VII.)}$$

applicable throughout a given region, and this exhibits the nature of the quaternion p of the present article. If there is no circulation at the boundary, so that we may put $\varpi = \nabla Q$ (where Q is a scalar function) in the surface integral, we have on replacing p by ∇Q in the identity already referred to

$$\nabla Q = \nabla\int\frac{\nabla'^2 Q'\mathrm{d}v'}{4\pi\mathrm{T}(\rho'-\rho)} - \nabla\int\frac{\mathrm{d}\nu'\nabla'Q'}{4\pi\mathrm{T}(\rho'-\rho)}; \quad \text{........(VIII.)}$$

also putting $p = Q$ in the first form of the same identity and introducing a new scalar function R,

$$R = Q - \int\frac{\nabla'^2 Q'\mathrm{d}v'}{4\pi\mathrm{T}(\rho'-\rho)} = -\int\frac{\mathrm{Sd}\nu'\nabla Q'}{4\pi\mathrm{T}(\rho'-\rho)} - \mathrm{S}\nabla\int\frac{\mathrm{d}\nu'Q'}{4\pi\mathrm{T}(\rho'-\rho)}. \quad \text{(IX.)}$$

Substituting for the surface integral from (VIII.) in (VII.) and attending to the definition of R in (IX.), we find

$$\varpi = \nabla P + \nabla\eta + \nabla R \quad \text{if } P = \int\frac{\mathrm{S}\nabla'\varpi'.\mathrm{d}v'}{4\pi\mathrm{T}(\rho'-\rho)}, \quad \eta = \int\frac{\mathrm{V}\nabla'\varpi'.\mathrm{d}v'}{4\pi\mathrm{T}(\rho'-\rho)}. \quad \text{(X.)}$$

Moreover R is given by (IX.) as a scalar surface integral depending on the values of $\mathrm{Sd}\nu\varpi$ and of Q over the boundary, and $\nabla^2 R = 0$ throughout the region. In this notation (II.) and (VI.) become

$$F = -\int \mathrm{S}\nabla\eta\mathrm{d}\rho + [P+R], \quad G = -\int \mathrm{Sd}\nu\nabla(P+R) - \int \mathrm{S}\eta\mathrm{d}\rho. \quad \text{(XI.)}$$

If $\eta = 0$, the distribution of the vectors ϖ is irrotational; if P is zero there is no divergence and the distribution is solenoidal;

if P and η both vanish, the distribution is irrotational and solenoidal.

If, as in Art. 130 (XVIII.), dn is the strength of a tube of vectors $V\nabla\varpi$ of cross-section $Td\omega$, and if $d\rho$ is along the tube, we have

$$V\nabla\varpi \,.\, dv = V\nabla\varpi \,.\, Td\omega Td\rho = d\rho dn \quad \text{because} \quad d\rho \parallel d\omega \parallel V\nabla\varpi.$$

If the tubes form closed rings and if $d\nu$ is the directed element of a surface bounded by a ring, we find (compare (X.))

$$\eta = \int \frac{d\rho' dn}{4\pi T(\rho'-\rho)} = \int dn V d\nu' \nabla' \,.\, \frac{1}{4\pi T(\rho'-\rho)} = V\nabla \int \frac{dn d\nu'}{4\pi T(\rho'-\rho)};$$

or again

$$\eta = (\nabla - S\nabla) \int \frac{dn d\nu'}{4\pi T(\rho'-\rho)} = \nabla \int \frac{dn d\nu'}{4\pi T(\rho'-\rho)} + \int \frac{\Omega dn}{4\pi}, \quad \text{........(XII.)}$$

where Ω is the solid angle subtended at the extremity of ρ by the closed ring of strength dn,

because $Sd\nu'\nabla T(\rho'-\rho)^{-1} = Sd\nu' U(\rho'-\rho) \,.\, T(\rho'-\rho)^{-2} = -d\Omega.$

(See Chap. VII., Ex. 22, p. 86.)

Hence at any point outside the vortex rings, *i.e.* at a point at which ρ' does not equal ρ, we have

$$\nabla\eta = \frac{1}{4\pi} \nabla \int \Omega dn, \quad \varpi = \nabla\left(P + \frac{1}{4\pi}\int \Omega dn + R\right). \quad \text{......(XIII.)}$$

This well-known transformation is due to the fact that under the supposed conditions a certain quaternion is reduced to zero by the operation of ∇.

ART. **132.** By means of the transformations

$$\rho S(\nabla)\varpi = \rho S\nabla\varpi - \varpi, \quad \rho V(\nabla)\varpi = \rho V\nabla\varpi - 2\varpi,$$

$$\rho S\rho(\nabla)\varpi = \rho S\rho\nabla\varpi + V\rho\varpi, \quad \rho V\rho V(\nabla)\varpi = \rho V\rho V\nabla\varpi - 3V\rho\varpi, \text{...(I.)}$$

which may be verified without difficulty, we obtain the transformations,

$$\int \varpi \,.\, dv = \int \rho S\nabla\varpi \,.\, dv - \int \rho S d\nu\varpi$$

$$= \tfrac{1}{2}\int \rho V\nabla\varpi \,.\, dv - \tfrac{1}{2}\int \rho V d\nu\varpi;$$

$$\int V\rho\varpi \,.\, dv = -\int \rho S\rho\nabla\varpi \,.\, dv + \int \rho S\rho d\nu\varpi$$

$$= \tfrac{1}{3}\int \rho V\rho V\nabla\varpi - \tfrac{1}{3}\int \rho V\rho V d\nu\varpi. \quad \text{..........(II.)}$$

Another transformation, likewise depending on the invariantal properties of ∇, is

$$\int S\omega\varpi \,.\, dv = \int S\rho\omega d\nu\varpi - \int (S\rho\omega\nabla'\varpi' + S\rho\omega'\nabla'\varpi) dv; \quad \text{...(III.)}$$

and by introducing ρ and ∇ into any relation it is generally possible to find a transformation analogous to these.

Ex. 1. The momentum and the moment of momentum with respect to the origin of vectors ρ of a portion of a continuous medium of density c, may be thrown into the forms

$$\lambda = \int c\sigma dv = \int \rho S\nabla(c\sigma)dv - \int c\rho S d\nu\sigma = \tfrac{1}{2}\int \rho V\nabla(c\sigma)\,.\,dv - \tfrac{1}{2}\int c\rho V d\nu\sigma,$$

$$\mu = \int cV\rho\sigma dv = -\int \rho S\rho\nabla(c\sigma)dv + \int c\rho S\rho d\nu\sigma = \tfrac{1}{3}\int \rho V\rho V\nabla(c\sigma)dv - \tfrac{1}{3}\int c\rho V\rho V d\nu\sigma\,;$$

and the kinetic energy of the portion may be represented by

$$T = \int \tfrac{1}{2}cT\sigma^2 dv = -\tfrac{1}{2}\int cS\rho\sigma d\nu\sigma + \int c(S\rho\sigma S\nabla\sigma + S\rho\sigma V\nabla\sigma)dv + \tfrac{1}{2}\int S\rho\sigma\nabla c\sigma dv.$$

(a) For an incompressible substance of uniform density, if $2\epsilon = V\nabla\sigma$,

$$\lambda = c\int \sigma dv = -c\int \rho S d\nu\sigma = c\int \rho\epsilon dv - \tfrac{1}{2}c\int \rho V d\nu\sigma,$$

$$\mu = c\int V\rho\sigma dv = -2c\int \rho S\rho\epsilon dv + c\int \rho S\rho d\nu\sigma = \tfrac{2}{3}c\int \rho V\rho\epsilon dv - \tfrac{1}{3}c\int \rho V\rho V d\nu\sigma,$$

$$T = \tfrac{1}{2}c\int T\sigma^2 dv = -\tfrac{1}{2}c\int S\rho\sigma d\nu\sigma + 2c\int S\rho\sigma\epsilon dv.$$

Ex. 2. In the notation of Art. 131, the kinetic energy may be expressed by

$$T = -\tfrac{1}{2}\int c(S\eta\sigma d\nu + (P+R)S\sigma d\nu) + \tfrac{1}{2}\int \dot{c}(P+R)dv - \tfrac{1}{2}\int S\eta\nabla(c\sigma)\,.\,dv\,;$$

and for an incompressible substance of uniform density,

$$T = -\tfrac{1}{2}c\int (S\eta\sigma d\nu + RS\sigma d\nu) - c\int S\eta\epsilon dv,$$

and the volume integral is

$$-c\int S\eta\epsilon dv = -c\int\int \frac{S\epsilon\epsilon' dv dv'}{2\pi T(\rho - \rho')}.$$

(vii) *Equations of motion of a deformable system.*

ART. **133.** For any system of particles the equations (compare Arts. 119 and 120, p. 194)

$$M\,.\,D_t\sigma = \lambda,\ \ D_t\int V\tau\dot{\tau}\,.\,dm = \mu \quad \ldots\ldots\ldots\ldots\ldots\ldots (\text{I.})$$

are independent of the mutual reactions of the particles composing the system, M being the total mass, σ the velocity of the centre of the mass, τ the vector from the centre of mass to the particle dm, λ the resultant force and μ the resultant couple referred to the centre of mass.

Suppose the system of particles to compose a definite portion of a distribution of matter, and let each particle dm be acted on by a force ξdm and a couple ηdm due to external causes. In addition the portion of matter is subject to the interaction between it and the rest of the matter. The forces of the interaction on the portion may be supposed to be the resultant of a number of forces $\Phi d\nu$ acting at each point of the boundary of the portion, and $\Phi d\nu$ is a linear function of the *tensor* of $d\nu$—the vector element of the surface. Moreover if c is the density, we have $dm = c dv$, where dv is an element of the volume. The equation (I.) therefore may be replaced by

$$D_t\sigma \,.\, \int c dv = \int c\xi dv + \int \Phi d\nu \,; \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(II.)}$$

and $$D_t \,.\, \int V\tau\dot{\tau} \,.\, c dv = \int c(\eta + V\tau\xi) dv + \int V\tau\Phi d\nu ; \ldots\ldots\ldots \text{(III.)}$$

and the volume integrals are taken throughout the selected portion while the surface integrals are taken over its boundary.

When we take the portion of matter to be small, the volume integrals in (II.) are ultimately of the third order of small quantities and the surface integral is of the second order. Provided therefore $D_t\sigma$ is not excessively large for very small portions and provided $\Phi d\nu$ is a continuous function of the vector-element of surface $d\nu$, the surface integral must vanish independently of the volume integrals when the dimensions of the portion are greatly reduced; and if the portion is taken to be a tetrahedron whose vector faces are proportional to α, β, γ and δ, we see that the function $\Phi d\nu$ at any point must satisfy the condition

$$\Phi(\alpha + \beta + \gamma) = \Phi\alpha + \Phi\beta + \Phi\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(IV.)}$$

for all vectors α, β and γ, because we have for the evanescent tetrahedron $\Phi\alpha + \Phi\beta + \Phi\gamma + \Phi\delta = 0$, where $\alpha + \beta + \gamma + \delta = 0$. Thus Φ is a linear and vector function. We may therefore apply the integration theorem of Art. 125, Ex. 2, and replace $\int \Phi d\nu$ in (II.) by the volume integral $\int \Phi\nabla \,.\, dv$, in which ∇ operates on Φ *in situ*. Thus we have

$$D_t\sigma \,.\, \int c dv = \int (c\xi + \Phi\nabla) \,.\, dv \,; \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(V.)}$$

and when we reduce the portion, we find in the limit

$$D_t\sigma = \xi + c^{-1} \,.\, \Phi\nabla, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(VI.)}$$

where $D_t\sigma$ is the acceleration of the centre of mass of a small portion of the matter.

Applying the same principles of continuity and of dimensions to (III.), and taking the portion of matter to be a small parallelepiped whose edges are parallel to α, β and γ, we find

$$-c\eta S\alpha\beta\gamma + V\alpha\Phi V\beta\gamma + V\beta\Phi V\gamma\alpha + V\gamma\Phi\alpha\beta = 0 \,;$$

or simply (Art. 67, Ex. 7, p. 97)

$$c\eta + 2\epsilon = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(VII.)}$$

where ϵ is the spin-vector of Φ, as we see more easily by putting i, j and k for α, β and γ. Provided there is no voluminal distribution of couple, the function Φ is self-conjugate.

The equation of continuity is

$$\dot{c} = S\nabla(c\sigma) \quad \text{or} \quad -cS\rho_1\rho_2\rho_3 = C, \text{ ....(VIII.)}$$

according as we use Euler's or Lagrange's method (Art. 129), and by Art. 128 (VI.) or (XIV.) we may replace (VI.) by

$$\frac{\partial^2\rho}{\partial t^2} = \xi + C^{-1}\left(\frac{\partial}{\partial u} \cdot \Phi V\rho_2\rho_3 + \frac{\partial}{\partial v} \cdot \Phi V\rho_3\rho_1 + \frac{\partial}{\partial w} \cdot \Phi V\rho_1\rho_2\right). \text{ ...(IX.)}$$

Ex. 1. Find the equation of motion for a perfect fluid.

[The force $\Phi d\nu$ on the boundary of a portion of the fluid is $-pd\nu$, where p is the pressure, remembering that $d\nu$ is outwardly directed. Hence the equation is $D_t\sigma = \xi - c^{-1}\nabla p$.]

Ex. 2. Integrating along a stream line, show that

$$\tfrac{1}{2}T\sigma^2 + \int S(\xi + c^{-1}\Phi\nabla)d\rho$$

is constant for an element of the matter, and find the integral in the case of a fluid acted on by conservative forces.

Ex. 3. When the forces acting on a perfect fluid are conservative, the circulation in any circuit moving with the fluid remains unchanged provided the density is a function of the pressure.

[We have $D_t\sigma = -\nabla(P + \int c^{-1}dp)$. See Art. 130, Ex. 1. An independent proof is easily obtained by Lagrange's method, which gives

$$D_tF = -D_t\int S\sigma d\rho = -\frac{\partial}{\partial t}\int S\frac{\partial\rho}{\partial t}d\rho = -\int\frac{\partial^2\rho}{\partial t^2}d\rho - \int\frac{\partial\rho}{\partial t}d\,\frac{\partial\rho}{\partial t},$$

and if this vanishes for all closed circuits $V\nabla D_t\sigma = 0$.]

Ex. 4. If $F = -\int S\sigma d\rho$, show that

$$D_tF = -\int S(\xi + c^{-1}\Phi\nabla)d\rho + \tfrac{1}{2}[T\sigma^2].$$

ART. **134**. To determine the nature of the stress-function Φ for a viscous fluid, we assume as usual that the stress consists of a hydrostatic pressure and of a part linear in the rate of distortion of the fluid, and that the stress-function is coaxial with the strain-function. In the notation of Art. 124, the strain-function is $\frac{1}{2}(\phi + \phi')$, and the general linear function coaxial with this function and linear in its coefficients is of the form $n(\phi + \phi') + n'm''$, where n and n' are constants and where $m''(= -S\nabla\sigma)$ is the first invariant of ϕ or ϕ' or $\frac{1}{2}(\phi + \phi')$. Consequently the stress-function is of the form

$$\Phi\alpha = -p\alpha + n(\phi + \phi')\alpha + n'm''\alpha, \text{ ....(I.)}$$

α being an arbitrary vector and p being a hydrostatic pressure.

The hydrostatic pressure is defined more particularly (with changed sign) to be the mean of the principal stresses, or

$$-3p = M'' = -\Sigma Si\Phi i = -3p + (2n + 3n')m''.$$

Hence the coefficients n and n' are connected by the relation

$$2n+3n'=0; \quad \text{.........................(II.)}$$

and finally in terms of ∇ (Art. 124, p. 211),

$$\Phi a=-pa-n(\mathrm{S}a\nabla\,.\,\sigma+\nabla\,.\,\mathrm{S}a\sigma)+\tfrac{2}{3}na\mathrm{S}\nabla\sigma. \quad \text{.........(III.)}$$

If n does not vary from point to point of the fluid, the equation of motion becomes

$$\mathrm{D}_t\sigma=\xi-c^{-1}\,.\,\nabla p-c^{-1}n(\nabla^2\sigma+\tfrac{1}{3}\nabla\mathrm{S}\nabla\sigma); \quad \text{........(IV.)}$$

otherwise if n varies, it must undergo operation by the ∇ which replaces a.

In like manner for an isotropic elastic solid, if θ is the displacement,

$$\Phi a=-n(\mathrm{S}a\nabla\,.\,\theta+\nabla\mathrm{S}a\theta)-n'a\mathrm{S}\nabla\theta, \quad \text{...............(V.)}$$

assuming that the stress function is coaxial with the strain-function and linear in its constituents. The equation of motion becomes

$$\mathrm{D}_t^2\theta=\xi-c^{-1}n\nabla^2\theta-c^{-1}(n+n')\nabla\,.\,\mathrm{S}\nabla\theta. \quad \text{...........(VI.)}$$

ART. **135.** The rate of change of kinetic energy of any finite portion of the matter is

$$\mathrm{D}_t\int\tfrac{1}{2}c\mathrm{T}\sigma^2\,.\,dv=\mathrm{D}_t\int\tfrac{1}{2}\mathrm{T}\sigma^2\,.\,dm$$

$$=-\int\mathrm{S}\sigma\mathrm{D}_t\sigma\,.\,dm=-\int\mathrm{S}\sigma_0(c\xi+\Phi\nabla)dv, \quad \text{.....(I.)}$$

and in the last integral ∇ operates on Φ but not on σ as indicated by the suffix. Because $\mathrm{S}\sigma\Phi\nabla=\mathrm{S}\sigma_0\Phi\nabla+\mathrm{S}\sigma\Phi_0\nabla$, where ∇ operates on the unsuffixed symbols, we may integrate by parts, and we find

$$\mathrm{D}_t\int\tfrac{1}{2}c\mathrm{T}\sigma^2\,.\,dv=-\int c\mathrm{S}\sigma\xi\,.\,dv+\int\mathrm{S}\sigma\Phi_0\nabla\,.\,dv-\int\mathrm{S}\sigma\Phi d\nu, \quad \text{...(II.)}$$

where $d\nu$ is an outwardly directed element of the boundary of the portion of matter.

For comparison we give the expression for the rate of change of kinetic energy in any region fixed in space. It is

$$\frac{d}{dt}\int\tfrac{1}{2}c\mathrm{T}\sigma^2\,.\,dv=\int\tfrac{1}{2}\dot{c}\mathrm{T}\sigma^2dv-\int c\mathrm{S}\sigma\dot{\sigma}\,.\,dv$$

$$=\int\tfrac{1}{2}\mathrm{T}\sigma_0{}^2\mathrm{S}\nabla(c\sigma)-c\mathrm{S}\sigma_0\nabla\mathrm{S}\sigma_0\sigma\}dv-\int\mathrm{S}\sigma_0(c\xi+\Phi\nabla)dv,$$

on making substitutions from the equations of continuity and of motion.

Now $-S\sigma_0\nabla S\sigma_0\sigma = +S\sigma_0\nabla \,.\, \frac{1}{2}T\sigma^2$ and the first integral changes at once into a surface integral so that

$$\frac{d}{dt}\int \tfrac{1}{2}cT\sigma^2 \,.\, dv$$

$$= \int \tfrac{1}{2}cT\sigma^2 \,.\, S\sigma d\nu - \int cS\sigma\xi \,.\, dv + \int S\sigma\Phi_0\nabla \,.\, dv - \int S\sigma\Phi d\nu, \ldots\text{(III.)}$$

transformation of the second part of the integral being as before. The difference between (II.) and (III.) is due to the influx of matter through the boundary.

The first integral in (II.) is due to the *activity* of the applied forces; the third is due to that of the surface stresses; the second, with sign changed, gives the rate at which energy is stored in the medium or dissipated.

ART. **136**. In the case of a viscous fluid, the rate of storage and waste of energy per unit volume is (Art. 134 (III.))

$$-S\sigma\Phi_0\nabla = pS\nabla\sigma + n(S\nabla\nabla'S\sigma\sigma' + S\nabla\sigma'S\nabla'\sigma) - \tfrac{2}{3}n(S\nabla\sigma)^2. \ldots\text{(I.)}$$

By the aid of the equation of continuity (Art. 133 (VIII.)) the term in p may be replaced by

$$pD_t \log c = D_t\int pc^{-1}dc = -D_t\int pb^{-1}db, \ldots\ldots\ldots\ldots\text{(II.)}$$

where b is the *bulkiness*, the reciprocal of the density; and for a given mass the rate of change of the intrinsic energy is

$$\int pS\nabla\sigma \,.\, dv = -\int pD_t b dm = -D_t\int dm\int pdb. \ldots\ldots\text{(III.)}$$

The part quadratic in σ is called by Lord Rayleigh the *dissipation function*, and it measures the rate at which energy per unit volume is wasted by the viscosity. This depends on the distortion, and it is expressible in terms of the elongations e_1, e_2 and e_3—the latent roots of the function $\phi_0 = \frac{1}{2}(\phi + \phi')$ of Art. 124.

The invariant m' of ϕ is (Art. 124 (VII.), p. 213)

$$m' = -\tfrac{1}{2}SV\nabla\nabla'V\sigma\sigma' = \tfrac{1}{2}S\nabla\sigma S\nabla'\sigma' - \tfrac{1}{2}S\nabla\sigma'S\nabla'\sigma;$$

also we have

$$4\epsilon^2 = V\nabla\sigma^2 = SV\nabla\sigma V\nabla'\sigma' = S\nabla\sigma'S\nabla'\sigma - S\nabla\nabla'S\sigma\sigma';$$

and from these two expressions we get

$$S\nabla\nabla'S\sigma\sigma' + S\nabla\sigma'S\nabla'\sigma = -4\epsilon^2 - 4m' + 2m''^2$$

since $m'' = -S\nabla\sigma = -S\nabla'\sigma'$. Thus

$$2F = n(S\nabla\nabla'S\sigma\sigma' + S\nabla\sigma'S\nabla'\sigma - \tfrac{2}{3}S\nabla\sigma^2) = \tfrac{4}{3}n(m''^2 - 3m' - 3\epsilon^2). \text{ (IV.)}$$

But (Art. 68, p. 98) the invariants of ϕ_0 are

$$m'' = e_1 + e_2 + e_3 \text{ and } m'' + \epsilon^2 = e_2 e_3 + e_3 e_1 + e_1 e_2,$$

and therefore

$$F = \tfrac{1}{3} n\{(e_2 - e_3)^2 + (e_3 - e_1)^2 + (e_1 - e_2)^2\}. \quad \text{(v.)}$$

Hence it follows that if the dissipation function vanishes the distortion of any element must be a uniform dilatation or contraction, for the conditions are

$$e_1 = e_2 = e_3. \quad \text{(vi.)}$$

Ex. For a dynamical system consisting of a solid and a fluid, the momentum and the moment of momentum of the system referred to the centre of mass of the solid are given by

$$\lambda = Mv + \int \sigma \mathrm{d}m, \quad \mu = \phi\omega + \int V\rho\sigma \mathrm{d}m,$$

ω being the angular velocity of the solid, v the velocity of its centre of mass, $\phi\omega$ the moment of momentum of the solid, ρ a vector from the centre of mass of the solid to an element dm of the fluid which is moving with velocity σ.

(a) In general (μ, λ) is the resultant wrench of the system of impulses which would generate the motion, and if the motion of the fluid is due to that of the solid, λ and μ are functions of v and ω; but if the motion can be generated by applying the wrench to the solid, it follows from Newton's law of the composition of velocities that λ and μ are linear functions of v and ω, or that (p. 208, Ex. 12)

$$\lambda = \phi_1 v + \phi_2 \omega, \quad \mu = \phi_2' v + \phi_3 \omega,$$

where ϕ_1, ϕ_2, ϕ_2' and ϕ_3 are four linear vector functions.

(b) The work done in altering v and ω to $v + \mathrm{d}v$ and $\omega + \mathrm{d}\omega$ is

$$\mathrm{d}W = -S\lambda \mathrm{d}v - S\mu \mathrm{d}\omega;$$

and if the dynamical system is conservative, so that dW is the differential of a function W of v and ω, the functions ϕ_1 and ϕ_3 must be self-conjugate and ϕ_2' must be the conjugate of ϕ_2.

(c) In the case of a perfect fluid, the velocity generated in this way must be irrotational, and assuming that σ, as well as λ and μ, is a linear function of v and ω, we must have

$$\sigma = \nabla(Sv\theta + S\omega\zeta),$$

where θ and ζ are vector functions of the vector ρ.

(d) In the case of a solid moving in an infinite liquid of uniform density, or of a solid containing a cavity filled with liquid, the functions θ and ζ must satisfy

$$\nabla^2\theta = 0, \quad \nabla^2\zeta = 0$$

throughout the liquid. And at the surface of the solid in contact with the liquid

$$S(v + V\omega\rho)\mathrm{d}\nu = S\mathrm{d}\nu\nabla . (Sv\theta + S\omega\zeta),$$

so that θ and ζ must satisfy the surface conditions

$$\mathrm{d}\nu = S\mathrm{d}\nu\nabla . \theta, \quad V\rho\mathrm{d}\nu = S\mathrm{d}\nu\nabla . \zeta.$$

(*e*) In this case we may replace the expressions for λ and μ by

$$\lambda = Mv + c\int d\nu(Sv\theta + S\omega\zeta), \quad \mu = \phi\omega + c\int V\rho d\nu(Sv\theta + S\omega\zeta);$$

and by the aid of the conditions which θ and ζ satisfy, it may be shown that

$$\int d\nu S\alpha\theta = \int S\nabla\nabla' . \theta S\alpha\theta' . dv, \quad \int V\rho d\nu S\alpha\zeta = \int S\nabla\nabla' . \zeta S\alpha\zeta' . dv,$$

$$\int V\rho d\nu S\alpha\theta = \int S\nabla\nabla' . \zeta S\alpha\theta' . dv, \qquad \int d\nu S\alpha\zeta = \int S\nabla\nabla' . \theta S\alpha\zeta' . dv,$$

so that the conditions (*b*) are satisfied. Also the functions ϕ_1, ϕ_2 and ϕ_3 depend on the nature of the solid and on the density of the liquid, and they are invariably related to the solid.

(*f*) If the solid is acted on by an applied wrench (η, ξ) referred to its centre of mass, the equations of motion, analogous to Euler's equation for a rigid body, are

$$\phi_1\dot{v} + \phi_2\dot{\omega} + V\omega(\phi_1 v + \phi_2\omega) = \xi,$$

$$\phi_2'\dot{v} + \phi_3\dot{\omega} + V\omega(\phi_2' v + \phi_3\omega) + Vv(\phi_1 v + \phi_2\omega) = \eta,$$

the second equation being obtained by expressing that the rate of change of the moment of momentum $(\mu + V\gamma\lambda)$ with respect to a fixed point is equal to the moment of the applied forces $(\eta + V\gamma\xi)$ with respect to that point.

(*g*) When there are no applied forces obtain and interpret the integrals

$$T(\phi_1 v + \phi_2\omega) = \text{const.}, \quad S(\phi_1 v + \phi_2\omega)(\phi_2' v + \phi_3\omega) = \text{const.},$$

$$Sv\phi_1 v + 2Sv\phi_2\omega + S\omega\phi_3\omega = \text{const.}$$

(*h*) When the linear momentum is constantly zero,

$$(\phi_3 - \phi_2'\phi_1^{-1}\phi_2)\dot{\omega} + V\omega(\phi_3 - \phi_2'\phi_1^{-1}\phi_2)\omega = \eta, \quad v = -\phi_1^{-1}\phi_2\omega,$$

and the angular velocity is that of a certain solid moving round a fixed point under the action of the couple η.

(*i*) For a steady motion of translation under no forces $Vv\phi_1 v = 0$; and in general for steady motion when ω does not vanish

$$v = -\phi_1^{-1}(\phi_2 + x)\omega, \quad V\omega[\phi_3 - (\phi_2' + x)\phi_1^{-1}(\phi_2 + x)]\omega = 0,$$

where x is a scalar. From this it follows that the axis of the screws of steady motion are parallel to edges of a sextic cone, and in general to each edge of the cone corresponds a single screw.

ART. **137**. In terms of the displacement θ, the equation for an elastic solid is (compare Art. 134 (VI.))

$$D_t^2\theta = \xi + c^{-1}\Phi\nabla, \quad \text{.........................(I.)}$$

the velocity σ being $\dot{\theta}$ and Φ being a self-conjugate function because there is no voluminal distribution of couple. The displacement θ is a function of the time and the position vector, and when the strain is small we may neglect the term $-S\dot{\theta}\nabla . \dot{\theta}$ in $D_t^2\theta$. We replace, in fact, $D_t^2\theta$ by the second derived of θ regarded as a function of t alone, that is by $\ddot{\theta}$. Observe that now ∇ is commutative in order of operation with the result of differentiating with respect to the time.

By Art. 135, the rate at which the forces work in storing and dissipating energy is the integral

$$\dot{W} = -\int S\dot{\theta}\Phi_0\nabla \, . \, dv \text{ ....(II.)}$$

taken throughout the body. By Hooke's law, stress is a linear function of strain. If the strain is multiplied by n, the function Φ is likewise multiplied by n. Suppose the strain to be gradually increased from zero so that at any stage the strain is n times the final amount where n is positive and less than unity. In this case (II.) becomes $\dot{W} = -\dot{n}n\int S\theta\Phi_0\nabla \, . \, dv$; and integrating between the limits 0 and 1, the total work done in producing the strain in this particular way is seen to be

$$W = -\tfrac{1}{2}\int S\theta\Phi_0\nabla \, . \, dv. \text{ ....(III.)}$$

If the work done is a function of the strain and not of the manner in which it has been produced, the function W is the *energy function*—a quadratic function of the strain, and the work done in altering the strain in any arbitrary manner is the difference of the values of the energy function corresponding to the final and the initial state.

When the energy function exists we see on comparison of (II.) and (III.) that in general for any two sets of strain answering to the displacements θ_1 and θ_2, we have

$$\int S\sigma_1\Phi_2\nabla_1 \, . \, dv = \int S\sigma_2\Phi_1\nabla_2 \, . \, dv. \text{ ....(IV.)}$$

In fact the theory is quite analogous to that of the linear function in the quadratic expression $S\rho\phi\rho$. If $dS\rho\phi\rho = 2Sd\rho\phi\rho$ the function ϕ must be self-conjugate, and $S\rho_1\phi\rho_2 = S\rho_2\phi\rho_1$ for all vectors. Conversely, if (IV.) holds good for all pairs of strains, the energy function exists.

The quaternion statement of Hooke's law is the function Φ is linear in the constituents of the self-conjugate function

$$\phi_0 a = \tfrac{1}{2}(\phi + \phi')a = -\tfrac{1}{2}(Sa\nabla \, . \, \theta - \nabla Sa\theta).$$

In other words, Φ is a linear function of ∇ and of θ, which is unchanged when θ and ∇ are interchanged, ∇ operating *in situ* on θ. Thus if a is an arbitrary vector free from the operation of ∇, Hooke's law is contained in the equation

$$\Phi a = \Theta(a, \nabla, \theta) = \Theta(a, \theta, \nabla), \text{ ....(V.)}$$

where Θ is a linear function of a, of ∇ and of θ.

In case the energy function exists

$$S\theta_1\Phi_2\nabla_1 = S\theta_1\Theta(\nabla_1, \nabla_2, \theta_2) = S\theta_1\Theta(\nabla_1, \theta_2, \nabla_2)$$
$$= S\theta_2\Phi_1\nabla_2 = S\theta_2\Theta(\nabla_2, \nabla_1, \theta_1) = S\theta_2\Theta(\nabla_2, \theta_1, \nabla_1). \quad \text{......(VI.)}$$

But we have already shown that Φ is self-conjugate, so we may equate the expressions (VI.) to the new expressions

$$S\nabla_1\Phi_2\theta_1 = S\nabla_1\Theta(\theta_1, \nabla_2, \theta_2) = S\nabla_1\Theta(\theta_1, \theta_2, \nabla_2)$$
$$= S\nabla_2\Phi_1\theta_2 = S\nabla_2\Theta(\theta_2, \nabla_1, \theta_1) = S\nabla_2\Theta(\theta_2, \theta_1, \nabla_1). \quad \text{......(VII.)}$$

We may sum up the whole matter in the following statement: writing for four arbitrary vectors

$$(\alpha, \beta, \gamma, \delta) = -S\alpha\Theta(\beta, \gamma, \delta), \quad \text{................(VIII.)}$$

the fact that Φ is self-conjugate allows us to interchange the positions of α and β; Hooke's law permits the interchange of γ and δ; the existence of the energy equation renders the pair α, β interchangeable with the pair γ, δ.

For any system of mutually rectangular unit vectors, i, j, k, we obtain from (V.) six self-conjugate vector functions (of α),

$$\Theta(\alpha, i, i), \Theta(\alpha, j, j), \Theta(\alpha, k, k), \Theta(\alpha, j, k), \Theta(\alpha, k, i), \Theta(\alpha, i, j), \quad \text{(IX.)}$$

with permission to interchange the positions of the second and third vectors. The thirty-six constituents of these functions are the thirty-six elastic constants in case the energy function does not exist. When the energy function does exist, the number of constants is at once reduced to twenty-one; three of the type (i, i, i, i); six (i, i, i, j); three (i, i, j, j); three (i, j, i, j); three (j, k, i, i) and three (j, i, k, i), using the notation indicated in (VIII.).

To exhibit clearly the meaning of these constants we shall employ a special notation for the strains. Let $\theta = iu + jv + kw$ and $\rho = ix + jy + kz$; let

$$s_{ii} = \frac{\partial u}{\partial x}, \text{ etc.}; \quad s_{ij} = s_{ji} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \quad \text{..................(X.)}$$

Then the stress across a directed area α arising from the strain s_{ii} is $\Theta(\alpha, i, i)s_{ii}$, and that arising from the strain s_{ij} is $\Theta(\alpha, i, j)s_{ij}$. The symbol $(ijki)$ represents the component of the stress across unit area j parallel to i due to unit strain of the type s_{ki}; and when the energy function exists this is equal to the component parallel to k of the stress across unit area i due to unit strain of the type s_{ij}.

Ex. 1. Show that the energy function is of the form

$$\tfrac{1}{2}\Sigma(iiii)s_{ii}^2 + \Sigma(iijj)s_{ii}s_{jj} + \tfrac{1}{2}\Sigma(ijij)s_{ij}^2 + \Sigma(ijjk)s_{ij}s_{jk}$$
$$+ \Sigma(iiij)s_{ii}s_{ij} + \Sigma(iijk)s_{ii}s_{jk}.$$

Ex. 2. Determine the reduction in the number of the elastic constants when the substance posseses a plane of symmetry.

(a) If the substance has two mutually rectangular planes of symmetry, the plane at right angles to both is a plane of symmetry.

[Reflection with respect to a plane of symmetry leaves the elastic properties unchanged. If k is normal to the plane, the constants whose symbols involve k an odd number of times must vanish. Thirteen of the twenty-one constants remain. When the substance has two planes of symmetry, at right angles to j and to k, only symbols of the types $(iiii)$, $(iijj)$ and $(ijij)$ remain, and hence the plane normal to i is also a plane of symmetry.]

Ex. 3. If the elastic constants referred to i, j, k remain unchanged when the axes of reference, i and j, are turned through two right angles round k, the plane perpendicular to k is a plane of symmetry.

[In this case change of i and j into $-i$ and $-j$ must leave the symbols unchanged.]

Ex. 4. Determine the conditions that the elastic constants may remain unchanged when i and j are rotated through a finite angle v round k.

[If α and β are the vectors obtained by turning i and j through an arbitrary angle u round k, the functions of u, $(kkk\alpha)$, $(k\alpha k\beta)$, etc., must be periodic functions of u for the period v or else reduce to constants. These functions can be expressed as sums of sines and cosines of u, $2u$, $3u$ and $4u$ together with constant terms. Hence the only admissible values of v are π, $\frac{2}{3}\pi$ or $\frac{1}{2}\pi$. In every case the symbols involving k three times must vanish. We have already considered rotation through two right angles. For rotation through $\frac{1}{2}\pi$, the symbols linear in k must also vanish, and changing i and j into $+j$ and $-i$ respectively must leave all symbols unaltered. Thus $(kkii)=(kkjj)$, $(kkij)=0$, etc., and $(iiij)+(jjji)=0$, $(iiii)=(jjjj)$. For rotation through $\frac{2}{3}\pi$ the functions of u independent of k or involving k twice must reduce to constants. We find in addition to the conditions satisfied for rotation through one right angle that $(iiij)=(jjji)=0$, $(iiii)=(iijj)+2(ijij)$. Expressing that $(k\alpha\alpha\alpha)$, $(k\alpha\alpha\beta)$ are functions of $\cos 3u$ and $\sin 3u$, we get $-(kiii)=(kjij)=(kijj)$, $-(kjjj)=(kiji)=(kjii)$. For rotation through an arbitrary angle the symbols linear in k must vanish and the conditions for $v=\frac{2}{3}\pi$ must hold.]

Ex. 5. When the energy function exists prove the existence of a self-conjugate function ϕ for which the relation

$$\Theta(\alpha, \beta, \gamma)-\Theta(\beta, \alpha, \gamma)=\mathrm{V}\,.\,\phi\mathrm{V}\alpha\beta\,.\,\gamma$$

is identically true.

(a) The axes of ϕ, when determinate, form a natural system of lines of reference, and where a plane of symmetry exists, it is normal to an axis.

[The function on the left is obviously a linear function of $\mathrm{V}\alpha\beta$. Operating by $\mathrm{S}\delta$ we have

$$(\delta\alpha\beta\gamma)-(\delta\beta\alpha\gamma)=(\beta\gamma\delta\alpha)-(\beta\delta\gamma\alpha)=-\mathrm{SV}\gamma\delta\phi\mathrm{V}\alpha\beta=-\mathrm{SV}\alpha\beta\phi\mathrm{V}\gamma\delta,$$

and as this is a symmetrical function of $\mathrm{V}\alpha\beta$ and of $\mathrm{V}\gamma\delta$ the self-conjugate character of ϕ is established.

For an arbitrary set of mutually rectangular axes, we have

$$\Theta(i', j', k')-\Theta(j', i', k')=\mathrm{V}\phi k'\,.\,k', \text{ etc.,}$$

whence it follows that if i, j and k are the axes of ϕ, the vectors are completely permutable in $\Theta(i, j, k)$, so that $(iijk)=(ijik)$, etc.

We easily find $-\mathrm{S}j'\phi k'=(i'i'j'k')-(i'j'i'k')$, $-\mathrm{S}i'\phi i'=(j'k'j'k')-(j'j'k'k')$, so that if k' is normal to a plane of symmetry it is an axis of ϕ.

If e_1, e_2, e_3 are the latent roots of ϕ we have in terms of the axes

$$e_1=(jkjk)-(jjkk),\ e_2=(kiki)-(kkii),\ e_3=(ijij)-(iijj).$$

Given the constants referred to axes i', j', k' we can on transformation to the axes of ϕ determine whether there are planes of symmetry or not.

In general putting $\rho=zk+r\alpha$, where $\alpha=i\cos u+j\sin u$, we have the expansion

$$(\rho\rho\rho\rho)=z^4(kkkk)+4z^3r(kkk\alpha)+2z^2r^2\{(kk\alpha\alpha)+2(k\alpha k\alpha)\}+4zr^3(k\alpha\alpha\alpha)+r^4(\alpha\alpha\alpha\alpha),$$

and when i, j and k are axes of ϕ we have also

$$(k\alpha\alpha\alpha)=(kiii)\cos^3u+3(kiij)\cos^2u\sin u+3(kijj)\cos u\sin^2u+(kjjj)\sin^3u$$

because the letters in a symbol involving i, j and k are completely permutable for this special set of axes. Hence it follows that a plane $z=0$ which is a plane of symmetry of the quartic $(\rho\rho\rho\rho)$ and of the quadric $S\rho\phi\rho$ is a plane of elastic symmetry. The coefficients of the powers of $\cos u$ and $\sin u$ in $(kkk\alpha)$ and in $(k\alpha\alpha\alpha)$ must then vanish, and by the special laws of interchange every coefficient of odd order in k vanishes.

Suppose now that the plane $Sj\rho=0$ or $u=0$ is a plane of symmetry. The coefficients of the powers of z must be functions of $\cos u$ alone. Thus

$$(\rho\rho\rho\rho)=z^4a+4z^3rb\cos u+6z^2r^2(c\cos 2u+c')+4zr^3(d\cos 3u+d'\cos u)$$
$$+r^4(e\cos 4u+e'\cos 2u+e'')$$

suppose. If the plane $u=v$ is also a plane of symmetry, this function must be independent of the sign when we put $u=v\pm w$, where w is arbitrary. Hence $\quad b\sin v=c\sin 2v=d\sin 3v=d'\sin v=e\sin 4v=e'\sin 2v=0,$

and unless the quartic is a surface of revolution, the only admissible values of v are $\frac{1}{2}\pi$, $\frac{1}{3}\pi$ and $\frac{1}{4}\pi$. Hence planes of elastic symmetry must intersect at angles of 90°, 60° or 45° if every plane through their intersection is not a plane of symmetry. Of course in the second and third cases, the quadric $S\rho\phi\rho$ is of revolution. There is no difficulty in writing down the elastic constants for each case.

Suppose two roots of ϕ to be equal so that there are indeterminate axes in the plane of i and j, and that it is required to find a natural system of lines of reference. We may equate to zero the derived with respect to u of the first of the coefficients $(kkk\alpha)$, $(kk\alpha\alpha)+2(k\alpha k\alpha)$, $(k\alpha\alpha\alpha)$, $(\alpha\alpha\alpha\alpha)$ which does not vanish. Determining u from such an equation we take $i\cos u+j\sin u$ and $j\cos u-i\sin u$ along with k as the natural axes of reference. The case in which ϕ reduces to a constant will be considered in the next example.]

Ex. 6. When the energy function exists,

$$\tfrac{1}{2}\nabla^2\,.\,\Theta(\rho,\rho,\rho)=\Sigma\Theta(\rho,i,i)+2\Sigma\Theta(i,i,\rho)=\phi_2\rho,$$

is a self-conjugate vector function invariantally related to the elastic structure.

[The function is invariantal because ∇^2 is an invariant operator independent of any particular choice of i, j and k. If a plane of symmetry exists, it is a principal plane of this function, because if k is normal to a plane of symmetry, $Si\phi_2k$ and $Sj\phi_2k$ both vanish, being of odd order in k. Therefore k is an axis of ϕ_2, and ϕ_2 and ϕ of the last example have a common axis.

In terms of the axes i, j and k of the last example, it is easy to see that

$$-Si\phi_2i=3\Sigma(ii\alpha\alpha)+2(e_2+e_3),\quad -Si\phi j=3\Sigma(ij\alpha\alpha),$$

where α stands for i, j and k in the summation.

The axes of this function may be used as natural axes of reference when the function ϕ of the last example reduces to a constant e. In this case for arbitrary axes, i, j and k are completely permutable in any symbol in which they all occur, and $(jkjk)=e+(jjkk)$, etc.]

ART. **138.** In the notation of the last Article, the equation of vibrations of an elastic solid, not acted on by voluminal forces, is

$$c\ddot{\theta}=\Theta(\nabla, \nabla, \theta), \quad \text{..........................(I.)}$$

where, as we have said, $\ddot{\theta}$ is the second partial derived, with respect to the time, of θ, which is a function of t and ρ.

Consider the propagation of a plane wave. If the vector v represents in magnitude and direction the *wave-velocity*, the equation of a wave-front is

$$u=t-S\frac{\rho}{v}, \quad \text{................................(II.)}$$

for this represents a plane moving at right angles to itself with velocity v. Over a wave-front, the displacement from the mean position is, by definition, the same at every point at any given time. In other words θ is a function of u and of t. Hence

$$\nabla\theta=-\nabla S\frac{\rho}{v}\cdot\frac{\partial\theta}{\partial u}=\frac{1}{v}\cdot\frac{\partial\theta}{\partial u},$$

and generally if $f\nabla$ is a homogeneous function of ∇ of order n,

$$f\nabla\,.\,\theta=f\left(\frac{1}{v}\right)\cdot\frac{\partial^n\theta}{\partial u^n}. \quad \text{......................(III.)}$$

In particular (I.) becomes for plane wave motion

$$c\ddot{\theta}=\Theta\left(\frac{1}{v}, \frac{1}{v}, \frac{\partial^2\theta}{\partial u^2}\right). \quad \text{.....................(IV.)}$$

If the wave is of permanent type, θ involves t only as involved in u, and if in addition the vibration is harmonic and of frequency p,

$$\ddot{\theta}=\frac{\partial^2\theta}{\partial u^2}=-p^2\theta. \quad \text{.........................(V.)}$$

In this case (IV.) becomes

$$\Theta(Uv, Uv, \theta)=c\theta Tv^2. \quad \text{....................(VI.)}$$

This shows that for a plane wave propagated in the direction Uv, the vibration θ is parallel to an axis of the linear vector function* $\Theta(Uv, Uv, \alpha)$, and that the velocity is the square root of the quotient of the corresponding latent root by the density. The solid admits of three plane-polarised waves propagated in the same direction with different velocities. The wave-velocity surface is determined by the equation

$$S\left\{\Theta\left(\frac{1}{v}, \frac{1}{v}, \alpha\right)-c\alpha\right\}\left\{\Theta\left(\frac{1}{v}, \frac{1}{v}, \beta\right)-c\beta\right\}\left\{\Theta\left(\frac{1}{v}, \frac{1}{v}, \gamma\right)-c\gamma\right\}=0, \quad \text{(VII.)}$$

which is equivalent to the latent cubic of the function

$$\Theta(Uv, Uv, \alpha).$$

---

*The function $\Theta(Uv, Uv, \alpha)$ is not one of the functions $\Theta(\alpha, i, i)$ of the last Article. The second and third vectors may be interchanged in these expressions, not the first and second.

When the energy function exists, the linear function

$$\Theta(\mathrm{U}\upsilon, \mathrm{U}\upsilon, \alpha)$$

is self-conjugate because we have by the law of interchanges (Art. 137 (VIII.)), $S\beta\Theta(\mathrm{U}\upsilon, \mathrm{U}\upsilon, \alpha) = S\alpha\Theta(\mathrm{U}\upsilon, \mathrm{U}\upsilon, \beta)$. In this case the vibrations θ_1, θ_2, θ_3 for any direction of wave propagation are mutually rectangular. Moreover, since the function W is essentially positive, the latent roots of the function Θ are positive as well as real, and there are therefore three real wave-velocities $\mathrm{U}\upsilon\mathrm{T}\upsilon_1$, $\mathrm{U}\upsilon\mathrm{T}\upsilon_2$ and $\mathrm{U}\upsilon\mathrm{T}\upsilon_3$ in any direction.

When a linear function has indeterminate axes, the ψ function of $\phi - g$ vanishes where g is the repeated root (Art. 66). The condition for indeterminate directions of vibration is therefore

$$\mathrm{V}\left\{\Theta\left(\frac{1}{\upsilon}, \frac{1}{\upsilon}, \alpha\right) - c\alpha\right\}\left\{\Theta\left(\frac{1}{\upsilon}, \frac{1}{\upsilon}, \beta\right) - c\beta\right\} = 0, \text{......(VIII.)}$$

where α and β arbitrary vectors.

This equation admits of a finite number of solutions (υ), which correspond to Hamilton's *internal conical refraction*. These vectors terminate at double points on the wave-velocity surface.

The *index-surface* (MacCullagh) or the *surface of wave-slowness* (Hamilton) is the inverse

$$S\{\Theta(\mu, \mu, \alpha) - c\alpha\}\{\Theta(\mu, \mu, \beta) - c\beta\}\{\Theta(\mu, \mu, \gamma) - c\gamma\} = 0 \text{ ...(IX.)}$$

of the wave-velocity surface (VII.), the vector μ being equal to $-\upsilon^{-1}$.

The *wave-surface*, or the surface of *ray-velocity*, is the envelope of the plane

$$S\frac{\rho}{\upsilon} = 1 \text{ or } S\mu\rho = -1, \text{ ....(X.)}$$

subject to the condition (VII.) or (IX.). That is, the wave-surface is the reciprocal of the index surface with respect to the unit sphere $\rho^2 + 1 = 0$; or it is the envelope of plane wave-fronts in unit time after passing through the origin; or it is the wave of the vibration propagated from the origin in unit time; or the vectors ρ which satisfy its equation represent in magnitude and direction the ray-velocities.

When the energy function exists a simple and remarkable expression may be found for the ray-velocity ρ in terms of μ and θ. The wave-surface may be expressed by elimination between

$$\Theta(\mu, \mu, \theta) = c\theta,\ d\Theta(\mu, \mu, \theta) = cd\theta,\ S\mu\rho + 1 = 0,\ S\rho d\mu = 0. \text{...(XI.)}$$

The second equation is in full

$$\Theta(d\mu, \mu, \theta) + \Theta(\mu, d\mu, \theta) + \Theta(\mu, \mu, d\theta) = cd\theta;$$

and operating on this by $S\theta$ and attending to the law of interchanges (Art. 137 (VIII.)),

$$2Sd\mu\Theta(\theta, \theta, \mu)+Sd\theta\Theta(\mu, \mu, \theta)=cS\theta d\theta;$$

and by (XI.) this reduces to

$$Sd\mu\Theta(\theta, \theta, \mu)=0.$$

Thus every $d\mu$ is perpendicular to $\Theta(\theta, \theta, \mu)$ and also to ρ, so that $\Theta(\theta, \theta, \mu)=x\rho$ where x is a scalar. Operating by $S\mu$ we find $-x=S\mu\Theta(\theta, \theta, \mu)=S\theta\Theta(\mu, \mu, \theta)=c\theta^2$, and therefore

$$\Theta(U\theta, U\theta, \mu)=c\rho. \qquad \text{(XII.)}$$

Further, if we operate on this by $S\mu$ and on the first of (XI.) by $S\theta$ we recover the relation $S\rho\mu+1=0$; so that all the relations connecting $U\theta$, μ and ρ are comprised in the two relations

$$\Theta(\mu, \mu, \theta)=c\theta, \quad \Theta(U\theta, U\theta, \mu)=c\rho. \qquad \text{(XIII.)}$$

(viii) *Electro-magnetic Theory.*

ART. **139**. The fundamental circuital laws of the electro-magnetic field are*

(I.) the circulation $\left(-\int S\eta d\rho\right)$ of the magnetic force (η) in any closed circuit is equal to the flux $\left(-\frac{1}{u}\int S\gamma d\nu\right)$ of the electric current (γ) through the circuit divided by the velocity of light (u) in free space;

(II.) the circulation, with changed sign, $\left(+\int S\epsilon d\rho\right)$ of the electric force (ϵ) in any closed circuit is equal to the flux $\left(-\frac{1}{u}\int S\gamma_{,}d\nu\right)$ of the magnetic current $(\gamma_{,})$ through the circuit divided by u.

These laws are symbolized by the relations

$$\int S\eta d\rho=\frac{1}{u}\int S\gamma d\nu, \quad \int S\epsilon d\rho=-\frac{1}{u}\int S\gamma_{,}d\nu; \qquad \text{(I.)}$$

and because it is implied that the fluxes of the vectors γ and $\gamma_{,}$ through the circuit are independent of any particular surface bounded by the circuit (Art. 130 (XV.)),

$$S\nabla\gamma=0, \quad S\nabla\gamma_{,}=0. \qquad \text{(II.)}$$

* We cannot delay to explain the units employed in this article. Full explanation will be found in the article by H. A. Lorentz on Maxwell's *Electromagnetische Theorie* in Bd. V_2, pp. 63-144, of the *Encyklopädie der mathematischen Wissenschaften.* These units are but slightly modified from Heaviside's rational units. Much use has been made of Lorentz's article and of Heaviside's work in the preparation of the account of the theory given in the text.

We proceed to define more particularly what is meant by the electric and magnetic current fluxes and by the electric and magnetic forces in these laws. The electric current flux through the circuit consists in general of three parts, the flux $(-\int S\iota d\nu)$ due to the conduction current (ι), the rate of change $(-D_t\int S\delta d\nu)$ of the electric displacement (δ) through the circuit, and the flux $(-\int eSv d\nu)$ due to the convection current (ev) where e is the density of electrification* carried through the circuit with velocity v. In like manner the magnetic current is due to the rate of change $(-D_t\int S\beta d\nu)$ of the magnetic induction (β) through the circuit, to a conduction current $(\iota_,)$ postulated by Heaviside, but probably non-existent, and to a convection current $(e_,v_,)$ where $e_,$ is the density of magnetification carried through the circuit with velocity $v_,$. On the whole the integral fluxes are

$$-\int S\gamma d\nu = -D_t\int S\delta d\nu - \int S\iota d\nu - \int eSv d\nu,$$

$$-\int S\gamma_, d\nu = -D_t\int S\beta d\nu - \int S\iota_, d\nu - \int e_,Sv_, d\nu. \quad \text{.........(III.)}$$

In the rate of change of the displacement through the circuit we must take account of the motion of the circuit which we suppose to move with the velocity σ, varying from point to point. We have therefore by Art. 129 (III.), p. 229.

$$\int S\gamma d\nu = \int S(\underset{-}{\dot{\delta}} + \iota + ev)d\nu, \quad \int S\gamma_, d\nu = \int S(\underset{-}{\dot{\beta}} + \iota_, + e_,v_,)d\nu, \quad \text{...(IV.)}$$

$$\text{where} \quad \underset{-}{\dot{\delta}} = \dot{\delta} - V\nabla V\sigma\delta - \sigma S\nabla\delta, \quad \underset{-}{\dot{\beta}} = \dot{\beta} - V\nabla V\sigma\beta - \sigma S\nabla\beta. \quad \text{.. ...(V.)}$$

Converting the line integrals in (I.) into surface integrals and expressing that the relations hold for every possible small circuit $d\nu$, we arrive at the differential equations of circuitation

$$V\nabla\eta = \frac{1}{u}(\underset{-}{\dot{\delta}} + \iota + ev), \quad V\nabla\epsilon = -\frac{1}{u}(\underset{-}{\dot{\beta}} + \iota_, + e_,v_,). \quad \text{........(VI.)}$$

We have not yet explained the meaning of the vectors ϵ and η. The total electric and magnetic forces at a point consist of impressed forces (ϵ_i and η_i) together with ϵ and η. Thus if ϵ_t and η_t are the total forces,

$$\epsilon_t = \epsilon + \epsilon_i, \quad \eta_t = \eta + \eta_i; \quad \text{.....................(VII.)}$$

* This is not electrification of the medium. It is due to charges of electricity carried by moving particles, for example.

and Lorentz further divides the impressed electric force into a part ϵ_{ic} co-operative with ϵ in producing the conduction current and a part ϵ_{id} co-operative with ϵ in producing the displacement. We shall write

$$\epsilon_t = \epsilon + \epsilon_{ic} + \epsilon_{id}, \quad \eta_t = \eta + \eta_{ic} + \eta_{ib}, \quad \text{.............. (VIII.)}$$

where the suffix i calls to mind that the force is impressed, c that it relates to conduction current, d to displacement and b to magnetic induction (β).

Expressing that the conduction currents are produced by the forces enumerated, we have

$$\iota = \Phi(\epsilon + \epsilon_{ic}), \quad \iota_{,} = \Phi_{,}(\eta + \eta_{ic}); \quad \text{.................. (IX.)}$$

and by Ohm's law in the case of isotropic media Φ is a scalar—the conductivity—and for anisotropic media Φ is a linear vector function. Similarly we suppose the postulated function $\Phi_{,}$ corresponding to the postulated magnetic conduction current $\gamma_{,}$ to be a linear vector function.

In like manner, expressing that the displacement (δ) and the induction (β) are due to the forces mentioned,

$$\delta = \phi(\epsilon + \epsilon_{id}), \quad \beta = \phi_{,}(\eta + \eta_{ib}). \quad \text{.................. (X.)}$$

The phenomena of *hysteresis* shows that ϕ and $\phi_{,}$ are not always linear functions of the forces, but we shall only consider the important case in which they are linear functions. For isotropic media, ϕ is the (scalar) dielectric constant and $\phi_{,}$ is the magnetic permeability.

Some little care is necessary in differentiating these expressions when the medium is in motion. Owing to the motion ϕ may change its value at a point fixed in space.

ART. **140**. The activity of the impressed electric and magnetic forces with reference to a small element of the medium of volume dv is

$$\begin{aligned} A_1 \mathrm{d}v &= -(\mathrm{S}\epsilon_{ic}\iota + \mathrm{S}\eta_{ic}\iota_{,} + \mathrm{S}\epsilon_{id}\dot{\underline{\delta}} + \mathrm{S}\eta_{ib}\dot{\underline{\beta}})\mathrm{d}v \\ &= -(\mathrm{S}\iota\Phi^{-1}\iota + \mathrm{S}\iota_{,}\Phi_{,}^{-1}\iota_{,} + \mathrm{S}\dot{\underline{\delta}}\phi^{-1}\delta + \mathrm{S}\dot{\underline{\beta}}\phi_{,}^{-1}\beta)\mathrm{d}v \\ &\quad + (\mathrm{S}\epsilon(\dot{\underline{\delta}} + \iota) + \mathrm{S}\eta(\dot{\underline{\beta}} + \iota_{,}))\mathrm{d}v, \quad \text{......................... (I.)} \end{aligned}$$

transformation being made by (IX.) and (X.) of the last article. Transforming again by (VI.) we find

$$\begin{aligned} A_1 \mathrm{d}v = -(&\mathrm{S}\iota\Phi^{-1}\iota + \mathrm{S}\iota_{,}\phi_{,}^{-1}\iota_{,} + \mathrm{S}\dot{\underline{\delta}}\phi^{-1}\delta + \mathrm{S}\dot{\underline{\beta}}\phi_{,}^{-1}\beta \\ &+ e\mathrm{S}\epsilon v + e_{,}\mathrm{S}\eta v_{,} + u\mathrm{S}\nabla \mathrm{V}\epsilon\eta) \,.\, \mathrm{d}v, \quad \text{........ (II.)} \end{aligned}$$

because we have $-\mathrm{S}\epsilon \mathrm{V}\nabla\eta + \mathrm{S}\eta \mathrm{V}\nabla\epsilon = \mathrm{S}\nabla \mathrm{V}\epsilon\eta$.

The electric and magnetic forces evoke mechanical forces, ξ per unit volume, and the stress $\Phi_s \mathrm{d}\nu$ across the directed element

$d\nu$. If the element moves with velocity σ the activity of these forces on the element is

$$A_2 dv = -(S\sigma\xi + S\sigma\Phi_s(\nabla))dv, \qquad \text{(III.)}$$

the last term, in which (∇) operates on Φ_s' and on σ *in situ*, being equal to the surface integral $-\int S\sigma\Phi_s d\nu$ over the element. The total activity

$$A dv = (A_1 + A_2)dv \qquad \text{(IV.)}$$

is equal to the rate of transfer of energy to the element.

The term

$$J = -S\iota\Phi^{-1}\iota - S\iota_{,}\Phi_{,}^{-1}\iota_{,} \qquad \text{(V.)}$$

is by Joule's law the rate of waste of energy per unit volume owing to the conversion of energy into heat by the resistance. The terms in this expression for the Joulian waste are analogous to the dissipation function of a viscous fluid. The term $eS\epsilon v + e_{,}S\eta v_{,}$ relates to the convection currents.

The work done in increasing the electric displacement by the amount $d\delta$ is

$$-S\epsilon_{td}d\delta = -S(\epsilon + \epsilon_{id})d\delta = -S\phi^{-1}\delta d\delta, \qquad \text{(VI.)}$$

where ϵ_{td} is the total electric force operative in producing the displacement. (Compare (VIII.) and (X.) of the last article.) Experiments on dielectrics show that an energy function exists, or in other words the work done is the differential of the function

$$W = -\tfrac{1}{2}S\delta\phi^{-1}\delta = -\tfrac{1}{2}S\epsilon_{td}\delta = -\tfrac{1}{2}S\epsilon_{td}\phi\epsilon_{td}, \qquad \text{(VII.)}$$

which represents the energy stored in unit volume of the medium and due to the electric force. From the existence of this energy function we infer that ϕ is self-conjugate. A similar result holds good for the magnetic induction, and the energy due to this cause is

$$W_{,} = -\tfrac{1}{2}S\beta\phi_{,}^{-1}\beta = -\tfrac{1}{2}S\eta_{tb}\beta = -\tfrac{1}{2}S\eta_{tb}\phi_{,}\eta_{tb}. \qquad \text{(VIII.)}$$

The energy stored in unit of volume due to electric and magnetic forces is the sum of W and $W_{,}$.

When the medium is at rest the total activity is (II.)

$$A dv = (J + \dot{W} + \dot{W}_{,} - eS\epsilon v - \epsilon_{,}S\eta v_{,} - uS\nabla V\epsilon\eta)dv, \qquad \text{(IX.)}$$

because in this case $\underset{\cdot}{\delta}$ and $\underset{\cdot}{\beta}$ must be replaced by $\dot{\delta}$ and $\dot{\beta}$. We have accounted for every term except the last. This by a process of exclusion represents the rate of radiation of energy from the small volume. It may be expressed as a surface integral,

$$-uS\nabla V\epsilon\eta \,.\, dv = -u\int S d\nu V\epsilon\eta, \qquad \text{(X.)}$$

and this is the total *outward* flux of the vector $uV\epsilon\eta$, the vector area $d\nu$ being outwardly directed as usual. This vector is the

Poynting vector—discovered independently by Professor Poynting and Mr. Oliver Heaviside. It represents in magnitude and direction the flux of radiated energy.

Granting that the same vector represents the energy flux when the medium is in motion, and there seems to be no adequate reason for doubt, the total activity is

$$A\,\mathrm{d}v = (J - e\mathrm{S}\epsilon v - e_{,}\mathrm{S}\eta v_{,} - u\mathrm{S}\nabla\mathrm{V}\epsilon\eta)\mathrm{d}v + \mathrm{D}_t(W\mathrm{d}v + W_{,}\mathrm{d}v), \quad \ldots(\text{XI.})$$

the last term being the rate of change of the energy stored in the element $\mathrm{d}v$ and due to electric and magnetic causes.

Equating this to the sum $(A_1 + A_2)\mathrm{d}v$ already obtained, we have

$$\mathrm{D}_t(W\mathrm{d}v + W_{,}\mathrm{d}v)$$
$$= -(\mathrm{S}\dot{\underline{\delta}}\phi^{-1}\delta + \mathrm{S}\dot{\underline{\beta}}\phi_{,}^{-1}\beta)\,.\,\mathrm{d}v - (\mathrm{S}\sigma\xi + \mathrm{S}\sigma\Phi_s(\nabla))\mathrm{d}v. \quad \ldots.(\text{XII.})$$

By Art. 129 (III.), p. 229, we find

$$\mathrm{D}_t(W\mathrm{d}v) = (\mathrm{D}_t W - W\mathrm{S}\nabla\sigma)\,.\,\mathrm{d}v$$
$$= -\mathrm{S}\mathrm{D}_t\delta\,.\,\phi^{-1}\delta\mathrm{d}v - \tfrac{1}{2}\mathrm{S}\delta\,.\,\mathrm{D}_t\phi^{-1}\,.\,\delta\,.\,\mathrm{d}v - W\mathrm{S}\nabla\sigma\,.\,\mathrm{d}v,$$

where $\mathrm{D}_t\phi^{-1}$ is the result of operating by D_t on the function ϕ^{-1}.

Further, by (IV.) of the same article,

$$\dot{\underline{\delta}} = \mathrm{D}_t\delta - \mathrm{V}\nabla'\mathrm{V}\sigma'\delta = \mathrm{D}_t\delta - \delta\mathrm{S}\nabla\sigma + \mathrm{S}\delta\nabla'\,.\,\sigma',$$

and therefore

$$-\mathrm{S}\dot{\underline{\delta}}\phi^{-1}\delta = -\mathrm{S}\mathrm{D}_t\delta\,.\,\phi^{-1}\delta - 2W\mathrm{S}\nabla\sigma - \mathrm{S}\delta\nabla'\mathrm{S}\sigma'\phi^{-1}\delta.$$

Hence equation (XII.) becomes

$$\mathrm{S}\sigma\xi + \mathrm{S}\sigma\Phi_s(\nabla) = \tfrac{1}{2}\mathrm{S}\delta\,.\,\mathrm{D}_t\phi^{-1}\,.\,\delta + \tfrac{1}{2}\mathrm{S}\delta\phi^{-1}\delta\mathrm{S}\nabla\sigma - \mathrm{S}\delta\nabla'\mathrm{S}\sigma'\phi^{-1}\delta$$
$$+ \tfrac{1}{2}\mathrm{S}\beta\,.\,\mathrm{D}_t\phi_{,}^{-1}\,.\,\beta + \tfrac{1}{2}\mathrm{S}\beta\phi_{,}^{-1}\beta\mathrm{S}\nabla\sigma - \mathrm{S}\beta\nabla'\mathrm{S}\sigma'\phi_{,}^{-1}\beta. \quad \ldots(\text{XIII.})$$

The first term on the right may be written

$$\tfrac{1}{2}\mathrm{S}\delta\,.\,\mathrm{d}_t\phi^{-1}\,.\,\delta - \tfrac{1}{2}\mathrm{S}\sigma\nabla\mathrm{S}\delta_0\phi^{-1}\delta_0,$$

where ∇ operates on ϕ^{-1} alone since we have generally

$$\mathrm{D}_t = \mathrm{d}_t - \mathrm{S}\sigma\nabla,$$

where d_t refers to the rate of change at a point fixed in space.

Consider now the term $\tfrac{1}{2}\mathrm{S}\delta\,.\,\mathrm{d}_t\phi^{-1}\,.\,\delta$, where $\mathrm{d}_t\phi^{-1}$ is the time rate of change of ϕ^{-1} at the extremity of the vector ρ

$$= ix + jy + kz.$$

This change depends on the rate of distortion and on the angular velocity of the anisotropic medium. In other words, it is a function of the operation of ∇ on σ. Let $\theta = iu + jv + kw$ be the displacement at the point so that $\sigma = i\dot{u} + j\dot{v} + k\dot{w}$, and let u_x, etc., denote the deriveds of u and w with respect to x, y and z. We have

$$-\tfrac{1}{2}S\delta\,.\,d_t\phi^{-1}\,.\,\delta=\frac{\partial W}{\partial u_x}\,.\,\dot{u}_x+\frac{\partial W}{\partial u_y}\,.\,\dot{u}_y+\frac{\partial W}{\partial v_x}\,.\,\dot{v}_x+\text{etc.}$$

$$=\frac{\partial W}{\partial u_x}\,.\,Si\nabla Si\sigma+\frac{\partial W}{\partial u_y}Sj\nabla Si\sigma+\frac{\partial W}{\partial v_x}Si\nabla Sj\sigma+\text{etc.}$$

$$=S\sigma\Theta\nabla$$

suppose, where ∇ operates on σ alone, and where

$$\Theta i=-i\frac{\partial W}{\partial u_x}-j\frac{\partial W}{\partial v_x}-k\frac{\partial W}{\partial w_x},\text{ etc. } \quad\text{(XIV.)}$$

Introducing this function Θ and an analogous function for the corresponding magnetic term, and accenting vectors σ operated on by ∇, we replace (XIII.) by

$$\begin{aligned}S\sigma(\xi+\Phi_s\nabla)+S\sigma'\Phi_s\nabla'&\\=S\sigma'(\Theta+\Theta_{,})\nabla'-\tfrac{1}{2}S\sigma\nabla\,.\,(S\delta_0\phi^{-1}\delta_0+S\beta_0\phi_{,}^{-1}\beta_0)&\\+\tfrac{1}{2}(S\delta\phi^{-1}\delta+S\beta\phi_{,}^{-1}\beta)S\nabla'\sigma'&\\-S\delta\nabla'S\sigma'\phi^{-1}\delta-S\beta\nabla'S\sigma'\phi_{,}^{-1}\beta,&\quad\text{(XV.)}\end{aligned}$$

Now this relation, or identity, is formally true for all velocities σ, and for all distortions and angular velocities $(\tfrac{1}{2}V\nabla\sigma)$; and by the principle of virtual velocities we equate corresponding terms of the relation. The symbolical statement of this principle is that the identity (XV.) remains true when we substitute for σ, ∇' and σ' any three arbitrary vectors λ, μ and ν. Hence

$$\xi+\Phi_s\nabla=-\tfrac{1}{2}\nabla(S\delta_0\phi^{-1}\delta_0+S\beta_0\phi_{,}^{-1}\beta_0),\quad\text{(XVI.)}$$

because $\rho_1=\rho_2$ if $S\lambda\rho_1=S\lambda\rho_2$ for all vectors λ; and again

$$\begin{aligned}\Phi_s\mu=(\Theta+\Theta_{,})\mu+\tfrac{1}{2}\mu(S\delta\phi^{-1}\delta+S\beta\phi_{,}^{-1}\beta)&\\-\phi^{-1}\delta S\delta\mu-\phi_{,}^{-1}\beta S\beta\mu,&\quad\text{(XVII.)}\end{aligned}$$

since $\phi_1=\phi_2$ if $S\nu\phi_1\mu=S\nu\phi_2\mu$ for all vectors μ and ν.

Replace μ in this expression by ∇ operating *in situ* on the various vectors, and we find

$$\begin{aligned}\Phi_s\nabla=(\Theta+\Theta_{,})\nabla+V\,.\,V\nabla\phi^{-1}\delta\,.\,\delta-\phi^{-1}\delta S\nabla\delta-\tfrac{1}{2}\nabla S\delta_0\phi^{-1}\delta_0&\\+V\,.\,V\nabla\phi_{,}^{-1}\beta\,.\,\beta-\phi_{,}^{-1}\beta S\nabla\beta-\tfrac{1}{2}\nabla S\beta_0\phi^{-1}\beta_0,&\quad\text{(XVIII.)}\end{aligned}$$

because

$$\tfrac{1}{2}\nabla S\delta\phi^{-1}\delta-\phi^{-1}\delta S\delta(\nabla)=\tfrac{1}{2}\nabla S\delta\phi^{-1}\delta-S\delta_0\nabla\,.\,\phi^{-1}\delta-\phi^{-1}\delta S\nabla\delta$$

and

$$\begin{aligned}V\,.\,V\nabla\phi^{-1}\delta\,.\,\delta&=\nabla S\delta_0\phi^{-1}\delta-S\delta_0\nabla\,.\,\phi^{-1}\delta\\&=\nabla\tfrac{1}{2}S\delta\phi^{-1}\delta+\nabla\tfrac{1}{2}S\delta_0\phi^{-1}\delta_0-S\delta_0\nabla\,.\,\phi^{-1}\delta.\end{aligned}$$

Thus we find for the mechanical force ξ,

$$\xi = -(\Theta + \Theta_{\prime})\nabla - \mathrm{V}\,.\,\mathrm{V}\nabla\phi^{-1}\delta\,.\,\delta + \phi^{-1}\delta\mathrm{S}\nabla\delta$$
$$-\mathrm{V}\,.\,\mathrm{V}\nabla\phi_{\prime}^{-1}\beta\,.\,\beta + \phi_{\prime}^{-1}\beta\mathrm{S}\nabla\beta. \;\ldots\ldots(\text{XIX.})$$

The stress across any small area is determined by (XVII.).

In general the terms in Θ and $\Theta_{\prime}$ are small, and we shall neglect them.

The stress across any small area due to the electric displacement is when we neglect Θ,

$$\Phi_e\mu = \tfrac{1}{2}\mu\mathrm{S}\delta\phi^{-1}\delta - \phi^{-1}\delta\mathrm{S}\delta\mu,$$

and if μ is parallel to $\phi^{-1}\delta$, we have

$$\Phi_e\mathrm{U}\phi^{-1}\delta = -\tfrac{1}{2}\mathrm{U}\phi^{-1}\delta\,.\,\mathrm{S}\delta\phi^{-1}\delta = +\mathrm{U}\phi^{-1}\delta\,.\,W,$$

while if μ is perpendicular to ϕ^{-1},

$$\Phi_e\mathrm{U}\mu = +\tfrac{1}{2}\mathrm{U}\mu\,.\,\mathrm{S}\delta\phi^{-1}\delta = -\mathrm{U}\mu\,.\,W.$$

Thus the stress consists of a tension along the lines $\mathrm{U}\phi^{-1}\delta$ and an equal pressure at right angles to these lines, numerically equal to the electric energy per unit volume. Similar results hold for the magnetic stress.

ART. **141.** When the circuit is at rest, and when there is no convection current the equations of circuitation become

$$\dot{\delta} + \iota = u\mathrm{V}\nabla\eta,\; \dot{\beta} = -u\mathrm{V}\nabla\epsilon, \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{I.})$$

when we put $\iota_{\prime} = 0$. When moreover the medium is at rest we have (Art. 139 (X.) and (IX.))

$$\dot{\delta} = \phi(\dot{\epsilon} + \dot{\epsilon}_{id}),\; \dot{\beta} = \phi_{\prime}(\dot{\eta} + \dot{\eta}_{ib}),\; \iota = \Phi(\epsilon + \epsilon_{ic})\,;\;\ldots\ldots\ldots\ldots(\text{II.})$$

and from these we obtain the equation

$$\phi(\ddot{\epsilon} + \ddot{\epsilon}_{id}) + \Phi(\dot{\epsilon} + \dot{\epsilon}_{ic}) + u^2\mathrm{V}\nabla\phi_{\prime}^{-1}\mathrm{V}\nabla\epsilon + u\mathrm{V}\nabla\dot{\eta}_{ib} = 0 \;\ldots(\text{III.})$$

which is explicit in the vector ϵ. Having determined ϵ from this equation, the impressed forces being known, we obtain δ, ι and $\dot{\beta}$ by direct operations on ϵ. The vectors $\dot{\eta}$ and $\mathrm{V}\nabla\eta$ are also expressible by direct operations in terms of ϵ.

There are two principal types of this equation. For a dielectric non-conductor Φ is zero, and the propagation of the disturbance is by waves. For a conductor incapable of storing electric energy, ϕ is zero and the propagation is by diffusion.

When there are no applied forces the equations (I.) and (II.) may be replaced by

$$\phi\dot{\epsilon} + \Phi\epsilon = u\mathrm{V}\nabla\eta,\; \phi_{\prime}\dot{\eta} + \Phi_{\prime}\eta = -u\mathrm{V}\nabla\epsilon\,;\;\ldots\ldots\ldots\ldots(\text{IV.})$$

and assuming $\quad \epsilon = \Sigma a_n\epsilon_n e^{b_n t},\; \eta = \Sigma a_n\eta_n e^{b_n t}, \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{V.})$

where the a and the b are constant scalars, the equations are identically satisfied provided ϵ_n and η_n satisfy the equations

$$b_n\phi\epsilon_n+\Phi\epsilon_n=u\mathrm{V}\nabla\eta_n,\ b_n\phi_{,}\eta_n+\Phi_{,}\eta_n=-u\mathrm{V}\nabla\epsilon_n \quad \text{......(VI.)}$$

and the boundary conditions. The scalars b must in general be determined by an equation arising from the boundary conditions. The scalars a depend on the initial state of the disturbance.

The particular solutions $\epsilon_n e^{b_n t}$, $\eta_n e^{b_n t}$, are the normal solutions, and for any two normal solutions we have

$$u\mathrm{S}\nabla\mathrm{V}\epsilon_1\eta_2+b_2\mathrm{S}\epsilon_1\phi\epsilon_2+b_1\mathrm{S}\eta_2\phi_{,}\eta_1+\mathrm{S}\epsilon_1\Phi\epsilon_2+\mathrm{S}\eta_2\Phi_{,}\eta_1=0, \quad \text{.........(VII.)}$$

because $\mathrm{S}\nabla\mathrm{V}\epsilon_1\eta_2=\mathrm{S}\eta_2\mathrm{V}\nabla'\epsilon_1'-\mathrm{S}\epsilon_1\mathrm{V}\nabla'\eta_2'$. Integrating throughout the medium and converting a volume integral into a surface integral we find,

$$u\int\mathrm{S}\mathrm{V}\epsilon_1\eta_2 d\nu+b_2\int\mathrm{S}\epsilon_1\phi\epsilon_2 dv+b_1\int\mathrm{S}\eta_2\phi_{,}\eta_1 dv+\int\mathrm{S}\epsilon_1\Phi\epsilon_2 dv+\int\mathrm{S}\eta_2\Phi_{,}\eta_1 dv=0\,;$$

$$u\int\mathrm{S}\mathrm{V}\epsilon_2\eta_1 d\nu+b_1\int\mathrm{S}\epsilon_2\phi\epsilon_1 dv+b_2\int\mathrm{S}\eta_1\phi_{,}\eta_2 dv+\int\mathrm{S}\epsilon_2\Phi\epsilon_1 dv+\int\mathrm{S}\eta_1\Phi_{,}\eta_2 dv=0, \quad \text{(VIII.)}$$

the second equation following by interchange of suffixes from the first.

If in either of these equations we replace b_1 and b_2 by conjugate complex expressions $b'\pm\sqrt{-1}b''$, and at the same time replace ϵ_1 and ϵ_2 by $\epsilon'\pm\sqrt{-1}\epsilon''$ and η_1 and η_2 by $\eta'\pm\sqrt{-1}\eta''$, the real part of the equations is

$$u\int\mathrm{S}(\mathrm{V}\epsilon'\eta'+\mathrm{V}\epsilon''\eta'')d\nu+b'\int(\mathrm{S}\epsilon'\phi\epsilon'+\mathrm{S}\epsilon''\phi\epsilon''+\mathrm{S}\eta'\phi_{,}\eta'+\mathrm{S}\eta''\phi_{,}\eta'')dv$$

$$+\int(\mathrm{S}\epsilon'\Phi\epsilon'+\mathrm{S}\epsilon''\Phi\epsilon''+\mathrm{S}\eta'\Phi_{,}\eta'+\mathrm{S}\eta''\Phi_{,}\eta'')dv=0, \quad \text{...(IX.)}$$

remembering in the reduction of this expression that ϕ and $\phi_{,}$ are self-conjugate (Art. 140 (VII.)). The surface integral is the total *inward* flux of energy across the boundary due to the disturbances ϵ', η' and ϵ'', η''. If no energy is communicated from outside the boundary, this is zero or negative —zero if no energy from inside escapes, and otherwise negative. The remaining integrals are all negative, the coefficient of b' being minus double the energy stored by the two distributions separately and the remaining integral being minus the energy wasted by conductive friction. Hence in any case b' cannot be positive. If there is no energy radiated and none dissipated, b' must be zero or else ϵ', ϵ'', η' and η'' must vanish so that there is no disturbance. On the whole then, the real parts of the scalars b are zero or negative when the medium receives no external energy; when in addition there is no dissipation and no radiation of energy across the boundary the real parts are zero, and in this case there are permanent oscillations within the medium, the scalars a being determined once for all by the initial conditions.

ART. **142**. We shall now give a sketch of the theory of the propagation of light in a crystalline medium adopting Clerk Maxwell's hypothesis. The medium being supposed non-conducting the functions Φ and $\Phi_{,}$ disappear, and the equations of a free vibration become

$$\dot{\delta}_0=\phi\dot{\epsilon}_0=u\mathrm{V}\nabla\eta_0,\ \dot{\beta}_0=\phi_{,}\dot{\eta}_0=-u\mathrm{V}\nabla\epsilon_0, \quad \text{...........(I.)}$$

when ϕ and $\phi_{,}$ are two self-conjugate functions which are constant if the properties of the medium are the same for the same directions at all points.*

* The suffixes 0 are employed in these equations as we shall have more to deal with the vectors ϵ and η defined in (II.).

Assuming for a plane wave (Art. 138, p. 247) that

$$\epsilon_0 = \epsilon \sin n\left(t - \mathrm{S}\frac{\rho}{v}\right), \quad \eta_0 = \eta \sin n\left(t - \mathrm{S}\frac{\rho}{v}\right), \quad \text{..........(II.)}$$

where v is the wave-velocity, we find on substitution in (I.)

$$\delta = \phi\epsilon = u\mathrm{V}v^{-1}\eta, \quad \beta = \phi_{,}\eta = -u\mathrm{V}v^{-1}\epsilon. \quad \text{...........(III.)}$$

From these we obtain among other relations

$$-w = \mathrm{S}\epsilon\delta = \mathrm{S}\epsilon\phi\epsilon = u\mathrm{S}\epsilon v^{-1}\eta = \mathrm{S}\eta\phi_{,}\eta = \mathrm{S}\eta\beta\,; \quad \text{.........(IV.)}$$

which show that the magnetic energy per unit volume is equal to the electric energy, for we have

$$W = \tfrac{1}{2}\mathrm{S}\epsilon_0\phi\epsilon_0 = \tfrac{1}{2}w \sin^2 n\left(t - \mathrm{S}\frac{\rho}{v}\right) = W_{,}. \quad \text{...........(V.)}$$

The total energy is $w \sin^2 n\left(t - \mathrm{S}\frac{\rho}{v}\right)$, and the mean energy is consequently $\frac{1}{2}w$.

Again if ρ represents the *ray-velocity* we have

$$\mathrm{S}\frac{\rho}{v} = 1, \quad \mathrm{S}\rho\,\mathrm{d}v^{-1} = 0 \quad \text{.......................(VI.)}$$

for all differentials $\mathrm{d}v$. Differentiating (III.)

$$\mathrm{d}\delta = \phi\mathrm{d}\epsilon = u\mathrm{V}(\mathrm{d}v^{-1}\,.\,\eta + v^{-1}\mathrm{d}\eta),$$
$$\mathrm{d}\beta = \phi_{,}\mathrm{d}\eta = -u\mathrm{V}(\mathrm{d}v^{-1}\,.\,\epsilon + v^{-1}\mathrm{d}\epsilon)\,; \quad \text{.........(VII.)}$$

operating by $\mathrm{S}\epsilon$ on the first, or by $\mathrm{S}\eta$ on the second, and attending to (III.), we find

$$\mathrm{S}\mathrm{d}v^{-1}\mathrm{V}\epsilon\eta = 0, \quad \text{..........................(VIII.)}$$

because by (IV.) $\mathrm{S}\epsilon\mathrm{d}\delta$ and $\mathrm{S}\beta\mathrm{d}\eta$ are each equal to $-\frac{1}{2}\mathrm{d}w$.

As this holds for all values of $\mathrm{d}v$ we must have ρ parallel to $\mathrm{V}\epsilon\eta$, and by (IV.) we find for the ray-velocity

$$\rho = \frac{u\mathrm{V}\epsilon\eta}{w}, \quad \text{.............................(IX.)}$$

and this, it should be noticed, is parallel to the Poynting Flux (Art. 140). Again it is easy to deduce from (III.) and (IV.) the expression for the wave-velocity (v)

$$v^{-1} = \frac{\mathrm{V}\beta\delta}{uw}, \text{ or } v = \frac{uw}{\mathrm{V}\beta\delta}. \quad \text{....................(X.)}$$

We have now enumerated six vectors depending on the propagation of the wave which are connected by the relations,

$$\left.\begin{array}{lll} \rho = \dfrac{u}{w}\mathrm{V}\epsilon\eta, & \delta = \dfrac{u}{\mathrm{T}v^2}\mathrm{V}\eta v, & \beta = \dfrac{u}{\mathrm{T}v^2}\mathrm{V}v\epsilon, \\ v = \dfrac{\mathrm{T}v^2}{uw}\mathrm{V}\delta\beta, & \epsilon = \dfrac{1}{u}\mathrm{V}\beta\rho, & \eta = \dfrac{1}{u}\mathrm{V}\rho\delta\,; \end{array}\right\} \text{........(XI.)}$$

and these vectors when drawn from a common origin pierce a concentric sphere in a pair of supplemental triangles.

When some one of the four vectors β, δ, ϵ and η is given, all the vectors can in general be determined subject to a choice of sign. If ϵ is given, we have $\delta = \phi\epsilon$, $w = -S\epsilon\delta$ and

$$\eta = \pm V\phi\epsilon\phi_{,}\epsilon\sqrt{\frac{-w}{SV\phi\epsilon\phi_{,}\epsilon\phi_{,}V\phi\epsilon\phi_{,}\epsilon}}\,; \quad \text{(XII.)}$$

for the equations give $S\eta\delta = 0$, $S\beta\epsilon = 0$, or $S\eta\phi\epsilon = 0$, $S\eta\phi_{,}\epsilon = 0$, and the suitable tensor is found by substituting $\eta = xV\phi\epsilon\phi_{,}\epsilon$ in $w = -S\eta\phi_{,}\eta$. Hence β, ρ and υ^{-1} can be found without ambiguity when the sign is selected. The case of exception is when ϵ (or η) is a solution of the equation

$$V\phi\alpha\phi_{,}\alpha = 0, \quad \text{(XIII.)}$$

or, in other words, an axis of the (generally non-conjugate) function $\phi^{-1}\phi_{,}$ or $\phi_{,}^{-1}\phi$.

When $U\upsilon$ or $U\rho$ is given, two independent values of the vectors can in general be found, and the solution corresponds to the splitting up of a wave or ray into two plane polarised waves travelling with a given direction for the wave- or the ray-velocity. Let us seek to determine δ and β from the second and third of (XI.) when $U\upsilon$ is given. We have

$$\delta = \frac{u}{T\upsilon}V\phi_{,}^{-1}\beta U\upsilon, \quad \beta = \frac{u}{T\upsilon}VU\upsilon\,.\,\phi^{-1}\delta, \quad \text{(XIV.)}$$

and from these, when we eliminate β and δ in turn, and introduce new linear functions ϕ_υ and $\phi_{,\upsilon}$, we find

$$\left.\begin{aligned} \phi_\upsilon\delta &= u^2V\,.\,\phi_{,}^{-1}VU\upsilon\,.\,\phi^{-1}\delta\,.\,U\upsilon = T\upsilon^2\,.\,\delta, \\ \phi_{,\upsilon}\beta &= u^2V\,.\,U\upsilon\,.\,\phi^{-1}V\phi_{,}^{-1}\beta U\upsilon = T\upsilon^2\,.\,\beta. \end{aligned}\right\} \quad \text{(XV.)}$$

Thus δ is an axis of the linear vector function denoted by ϕ_υ and $T\upsilon^2$ is the corresponding root, and because ϕ_υ has one zero root (corresponding to the axis $\phi U\upsilon$) there are only two finite latent roots or two values of the wave-velocity along the direction $U\upsilon$. That the functions ϕ_υ and $\phi_{,\upsilon}$ have the same latent roots appears from the fact that their latent cubics are equivalent to the equation in $T\upsilon^2$ obtained by eliminating β and δ from (XIV.). If $T\upsilon'^2$ is the second root of ϕ_υ and if δ' is the corresponding axis, we have

$$T\upsilon^2S\delta\phi^{-1}\delta' = u^2SVU\upsilon\phi^{-1}\delta'\phi_{,}^{-1}VU\upsilon\phi^{-1}\delta = T\upsilon'^2S\delta'\phi^{-1}\delta,$$

and therefore (by (XIV.)), since $T\upsilon^2$ is not generally equal to $T\upsilon'^2$,

$$S\delta\phi^{-1}\delta' = 0, \; S\beta\phi_{,}^{-1}\beta' = 0. \quad \text{(XVI.)}$$

But these conditions may be written in the form

$$S\delta\epsilon' = S\delta'\epsilon = S\beta\eta' = S\beta'\eta = 0, \quad \text{(XVII.)}$$

where ϵ', β' η', etc., correspond to δ'. Thus δ' is perpendicular to ϵ and υ, and therefore parallel to β by (XI.), and β' is parallel to δ. In fact we have

$$U\delta' = \pm U\beta,\ U\beta' = \mp U\delta, \qquad \text{(XVIII.)}$$

because $$U\upsilon = UV\delta\beta = UV\delta'\beta'.$$

Since δ and δ' satisfy the relations (compare (XVI.))

$$S\delta\phi^{-1}\delta' = 0,\ S\delta\phi_{,}^{-1}\delta' = 0,\ S\delta U\upsilon = 0,\ S\delta' U\upsilon = 0, \qquad \text{(XIX.)}$$

we easily find on putting $U\upsilon = UV\delta\delta' = V\delta\delta' : TV\delta\delta'$ in (XV.) that

$$u^2 S\delta'\phi_{,}^{-1}\delta' S\delta\phi^{-1}\delta = T\upsilon^2 TV\delta\delta'^2, \qquad \text{(XX.)}$$

and that

$$\upsilon = -\frac{u(S\delta'\phi_{,}^{-1}\delta' S\delta\phi^{-1}\delta)^{\frac{1}{2}}}{V\delta\delta'},\quad \upsilon' = -\frac{u(S\delta\phi_{,}^{-1}\delta S\delta'\phi^{-1}\delta')^{\frac{1}{2}}}{V\delta\delta'}. \qquad \text{(XXI.)}$$

This result leads to a simple construction. Let the quadrics

$$S\varpi\phi^{-1}\varpi = -1 \text{ and } S\varpi\phi_{,}^{-1}\varpi = -1 \qquad \text{(XXII.)}$$

be constructed. Then by (XIX.) δ and δ' are parallel to the pair of common conjugate radii in the central plane at right angles to the direction of the wave-velocity. Let ϖ and $\varpi_{,}$ be respectively the vector radii of the first and second quadrics parallel to δ, and let ϖ' and $\varpi_{,}'$ be those parallel to δ', then we have

$$\upsilon = -\frac{u}{V\varpi\varpi_{,}'},\quad \upsilon' = -\frac{u}{V\varpi_{,}\varpi'}, \qquad \text{(XXIII.)}$$

and from this construction everything relating to the wave can be determined. For the first set of signs in (XVIII.) we have

$$\left.\begin{array}{llll} \delta = \varpi\sqrt{w}, & \beta = \varpi_{,}'\sqrt{w}, & \delta' = \varpi'\sqrt{w'}, & \beta' = -\varpi_{,}\sqrt{w'}, \\ \epsilon = \phi^{-1}\varpi\sqrt{w}, & \eta = \phi_{,}^{-1}\varpi_{,}'\sqrt{w}, & \epsilon' = \phi^{-1}\varpi'\sqrt{w'}, & \eta' = -\phi_{,}^{-1}\varpi_{,}\sqrt{w'}, \\ \rho = uV\phi^{-1}\varpi\phi_{,}^{-1}\varpi_{,}', & & \rho' = uV\phi_{,}^{-1}\varpi_{,}\phi^{-1}\varpi', & \end{array}\right\} \text{(XXIV.)}$$

where w' is double the mean energy per unit volume for the second wave.* (Compare (IV.).)

From the fifth and sixth of equations (XI.) we have

$$\epsilon T\rho^{-1} = u^{-1}V\phi_{,}\eta U\rho,\ \eta T\rho^{-1} = u^{-1}VU\rho\,.\,\phi\epsilon\,; \qquad \text{(XXV.)}$$

and as in (XV.) we may write,

$$\left.\begin{array}{l} \phi_{\rho}\epsilon = u^{-2}V\phi_{,}VU\rho\,.\,\phi\epsilon\,.\,U\rho = \epsilon T\rho^{-2}, \\ \phi_{,\rho}\eta = u^{-2}VU\rho\,.\,\phi V\phi_{,}\eta U\rho = \eta T\rho^{-2}, \end{array}\right\} \qquad \text{(XXVI.)}$$

* Note that $\phi^{-1}\varpi$ has the same direction as the central perpendicular on the tangent plane to the quadric $S\varpi\phi^{-1}\varpi = -1$ at the extremity of ϖ and that its length is the reciprocal of that of the perpendicular.

and if we take ϵ and ϵ'' to be the two axes of ϕ_ρ corresponding to the two finite latent roots $T\rho^{-2}$ and $T\rho''^{-2}$ of the function, we find as before

$$S\epsilon\phi\epsilon''=0,\ \ S\epsilon\phi_{,}\epsilon''=0,\ \ S\eta\phi_{,}\eta''=0,\ \ S\eta\phi\eta''=0,$$

for it appears that $U\epsilon''=\pm U\eta$, $U\eta''=\mp U\epsilon$.

We can write down results analogous to (XXIII.) and (XXIV.) for the various vectors related to the waves whose ray-velocity is along a fixed direction $U\rho$.

We now return to equation (XII.), which we may write in the form

$$\eta=V\phi\epsilon\phi_{,}\epsilon\sqrt{\left(\frac{w}{m_{,}(w'w_{,}-w^2)}\right)}, \quad \text{(XXVII.)}$$

where $m_{,}$ is the third invariant of $\phi_{,}$ and where *

$$w=-S\epsilon\phi\epsilon,\ \ w_{,}=-S\epsilon\phi_{,}\epsilon,\ \ w'=-S\epsilon\phi\phi_{,}^{-1}\phi\epsilon, \quad \text{(XXVIII.)}$$

because we have

$$SV\phi\epsilon\phi_{,}\epsilon\phi_{,}V\phi\epsilon\phi_{,}\epsilon=m_{,}SV\phi\epsilon\phi_{,}\epsilon V\phi_{,}^{-1}\phi\epsilon\epsilon.$$

Expressing ρ and v^{-1} in terms of ϵ, by (XI.),

$$\left.\begin{aligned}\rho&=\frac{uV\epsilon V\phi\epsilon\phi_{,}\epsilon}{\sqrt{\{m_{,}w(w'w_{,}-w^2)\}}}=\frac{u(w_{,}\phi\epsilon-w\phi_{,}\epsilon)}{\sqrt{\{m_{,}w(w'w_{,}-w^2)\}}},\\ v^{-1}&=\frac{m_{,}V\,.\,V\phi_{,}^{-1}\phi\epsilon\epsilon\,.\,\phi\epsilon}{u\sqrt{\{m_{,}w(w'w_{,}-w^2)\}}}=\frac{w'\epsilon-w\phi_{,}^{-1}\phi\epsilon}{u\sqrt{\{m_{,}^{-1}w(w'w_{,}-w^2)\}}}.\end{aligned}\right\} \quad \text{(XXIX.)}$$

From these equations, on attending to (XXVIII.),

$$S\rho(w_{,}\phi-w\phi_{,})^{-1}\rho=0,\ \ S\rho\phi_{,}^{-1}\rho=-\frac{u^2w_{,}}{m_{,}w}; \quad \text{(XXX.)}$$

$$Sv^{-1}(w'\phi^{-1}-w\phi_{,}^{-1})^{-1}v^{-1}=0,\ \ Sv^{-1}\phi_{,}v^{-1}=-\frac{m_{,}w'}{u^2w}; \quad \text{(XXXI.)}$$

$$(w_{,}\phi-w\phi_{,})^{-1}\rho=m_{,}^{-1}u^2(w'-w\phi_{,}^{-1}\phi)^{-1}v^{-1}=\frac{u\epsilon}{\sqrt{\{m_{,}w(w'w_{,}-w^2)\}}}; \quad \text{(XXXII.)}$$

$$m_{,}(w'-w\phi\phi_{,}^{-1})\rho=u^2(w_{,}\phi-w\phi_{,})v^{-1}; \quad \text{(XXXIII.)}$$

$$m_{,}(\phi^{-1}w'-w\phi_{,}^{-1})\rho=u^2(w_{,}-w\phi^{-1}\phi_{,})v^{-1}; \quad \text{(XXXIV.)}$$

the last relations, which alone are likely to give trouble, being derived from (XXXII.) by operating with $(w'-w\phi_{,}^{-1}\phi)(w_{,}\phi_{,}^{-1}\phi-w)$ on both sides, remembering that in this the factors are commutative.

From (XXX.) and (XXXI.) the equations of the wave-velocity surface and of the ray-velocity surface may be written down, and equations (XXXIII.) and (XXXIV.) are suitable for investigating the cases of indeterminations which correspond to external and internal conical refraction.

Suppose, for example, that $w'=b^2w$, where b^2 is a latent root of the function $\phi\phi_{,}^{-1}$ and that β (not now the magnetic induction) is the corresponding axis while β' is the axis of the conjugate function $\phi_{,}^{-1}\phi$ corresponding to the same

* It should be noticed that w' has not here its recent meaning.

root. The equation (XXXIII.) fails to give a determinate value of ρ, and operating on it by $S\beta'$, we find

$$S\beta'\phi v^{-1}=0, \quad\text{.................................(XXXV.)}$$

since $\phi\beta'=b^2\phi_{,}\beta'$. Two other equations for v are obtained by putting $w'=b^2w$ in (XXXI.), and these are

$$Sv^{-1}(b^2\phi^{-1}-\phi_{,}^{-1})^{-1}v^{-1}=0,\quad Sv^{-1}\phi_{,}v^{-1}=-m_{,}b^2u^{-2}\,; \quad\text{.........(XXXVI.)}$$

and from these three equations we find four values of v^{-1}, say $\pm v_1^{-1}$ and $\pm v_2^{-1}$. Substituting the value v^{-1} in (XXXIII.) and replacing $w_{,}$ by its value in terms of ρ by (XXX.), we get

$$m_{,}(b^2-\phi\phi_{,}^{-1})\rho+m_{,}\phi v_1^{-1}S\rho\phi_{,}^{-1}\rho+u^2\phi_{,}v_1^{-1}=0, \quad\text{........(XXXVII.)}$$

and this equation represents a plane conic. For we have seen (XXXV.) that each vector in this expression is perpendicular to β', so that if α' and γ' are the remaining axes of $\phi_{,}^{-1}\phi$ corresponding to the latent roots a^2 and c^2, the equation is equivalent to the pair

$$\left.\begin{aligned} m_{,}(b^2-a^2)S\alpha'\rho+S\alpha'\phi v_1^{-1}(m_{,}S\rho\phi_{,}^{-1}\rho+a^2u^2)=0,\\ m_{,}(b^2-c^2)S\gamma'\rho+S\gamma'\phi v_1^{-1}(m_{,}S\rho\phi_{,}^{-1}\rho+c^2u^2)=0.\end{aligned}\right\} \quad\text{.....(XXXVIII.)}$$

In order to calculate in the most explicit manner the vectors v_1^{-1}, etc., we may by Art. 71, p. 100, reduce the functions $\phi_{,}$ and ϕ to the trinomial forms

$$\phi_{,}\lambda=-\Sigma\alpha S\alpha\lambda,\quad \phi\lambda=-\Sigma a^2\alpha S\alpha\lambda,\quad \phi_{,}^{-1}\lambda=-\Sigma\alpha'S\alpha'\lambda,\quad \phi^{-1}\lambda=-\Sigma a^{-2}\alpha'S\alpha'\lambda,$$

where identically $\lambda=-\Sigma\alpha S\alpha'\lambda=-\Sigma\alpha'S\alpha\lambda.$

Putting $v^{-1}=\alpha'p+\beta'q+\gamma'r$, equation (XXXV.) becomes $q=0$, while (XXXVI.) reduces to $p^2(c^{-2}-b^{-2})=r^2(b^{-2}-a^{-2})$ and $p^2+r^2=m_{,}b^2u^{-2}$, and we finally get for the four vectors

$$v^{-1}=\frac{\sqrt{m_{,}}\,.\,ac}{u}\left(\pm\frac{\alpha'}{a}\sqrt{\frac{a^2-b^2}{a^2-c^2}}\pm\frac{\gamma'}{c}\sqrt{\frac{b^2-c^2}{a^2-c^2}}\right). \quad\text{.........(XXXIX.)}$$

Again, taking $\rho=\alpha x+\beta y+\gamma z$, and substituting in (XXXVII.), we find a simple expression

$$m_{,}\{(b^2-a^2)ax+(b^2-c^2)\gamma z\}-m_{,}(aa^2p+\gamma c^2r)(x^2+y^2+z^2)+(ap+\gamma r)u^2=0 \quad\text{(XL.)}$$

for the equation of the conic traced out by the extremity of ρ. We notice that $m_{,}=S\alpha\beta\gamma^2$.

In order to obtain more explicit forms for the equations of the wave-surface and the wave-velocity surface, we note that the first equation (XXX.) expands into

$$w_{,}^2S\rho\psi\rho-w_{,}wS\rho\Psi\rho+w^2S\rho\psi_{,}\rho=0,$$

where ψ and $\psi_{,}$ are Hamilton's auxiliary functions and where

$$\Psi V\lambda\mu=V\phi\lambda\phi_{,}\mu+V\phi_{,}\lambda\phi\mu.$$

By the aid of the second equation (XXX.) this becomes

$$S\rho\psi\rho S\rho\psi_{,}\rho+u^2S\rho\Psi\rho+u^4=0. \quad\text{............................(XLI.)}$$

In like manner

$$Sv^{-1}\psi^{-1}v^{-1}Sv^{-1}\psi_{,}^{-1}v^{-1}+u^{-2}Sv^{-1}\Psi_{-1}v^{-1}+u^{-4}=0 \quad\text{..........(XLII.)}$$

is the equation of the wave-velocity surface, where

$$\Psi_{-1}V\lambda\mu=V\phi^{-1}\lambda\phi_{,}^{-1}\mu+V\phi_{,}^{-1}\lambda\phi^{-1}\mu.$$

Other forms may be given to the equation of the wave-surface such as

$$m_{,}^{2}\Sigma S\alpha'\rho^{2}\Sigma b^{2}c^{2}S\alpha'\rho^{2}-u^{2}m_{,}\Sigma(b^{2}+c^{2})S\alpha'\rho^{2}+u^{4}=0,$$

derived from (XLI.), and

$$\Sigma\frac{S\alpha'\rho^{2}}{m_{,}a^{2}\Sigma S\alpha'\rho^{2}-u^{2}}=0,$$

derived from (XXX.) by the aid of the trinomial expressions for the functions, but in problems treated by quaternions it is frequently preferable to deal directly with vector expressions rather than with the scalar equations of surfaces obtained by eliminating certain quantities from the vector equations.

Ex. Show that the wave-surface may be derived from a Fresnel's wave-surface by a pure strain.

[Put $\psi_{,}^{\frac{1}{2}}\rho=\rho'$ and $S\rho(w_{,}\phi-w\phi_{,})^{-1}\rho=S\rho'(w_{,}\psi_{,}^{\frac{1}{2}}\phi\psi_{,}^{\frac{1}{2}}-wm_{,})^{-1}\rho'$, also $u^{2}w_{,}=-wS\rho\psi_{,}\rho=wT\rho'^{2}$, etc.]

CHAPTER XVII.

PROJECTIVE GEOMETRY.

ART. **143**. There are several interpretations which may be assigned to a quaternion and which we have not yet explained. We now propose to show that a quaternion is capable of representing a definite point loaded with a definite weight or mass, and throughout this chapter we shall speak rather indifferently of quaternions or of points.*

In the identity

$$q = Sq \,.\left(1+\frac{Vq}{Sq}\right) = Sq \,.\,(1+OQ) \quad \text{if} \quad OQ = \frac{Vq}{Sq}, \qquad \text{(I.)}$$

it is manifest that the point Q at the extremity of the vector OQ drawn from an assumed origin is determined when the quaternion q is given, and that Sq is also determined. We regard Sq as a weight or a mass concentrated at the point. We shall sometimes use capital letters concurrently with small letters,

$$q = Q\,.\,Sq, \quad Q = 1+OQ, \qquad \text{(II.)}$$

to denote points of unit weight, or unit points, so that $Q\,.\,w$ denotes the point Q weighted with w. Thus $SQ=1$, $VQ=OQ$.

The difference of two unit points is the vector joining them,

$$Q-P = 1+OQ-(1+OP) = OQ-OP-PQ; \qquad \text{(III.)}$$

and the origin is the *scalar* point

$$O = 1. \qquad \text{(IV.)}$$

A vector represents the point at infinity along its direction, as appears by allowing Sq to diminish indefinitely in (I.) while Vq remains constant, for OQ will then increase indefinitely in length, so that at last Vq represents the point at infinity in its direction.

* See *Trans. R.I.A.*, vol. xxxii., and *Phil. Trans.*, vol. 201, pt. viii. I regret that at the time of publication of these papers I was not acquainted with an able memoir by Dr. James Byrnie Shaw (*American Journal of Mathematics*, vol. xix., pp. 193-216), in which somewhat similar results are obtained.

The relation

$$p+q+r=(\mathrm{S}p+\mathrm{V}p)+(\mathrm{S}q+\mathrm{V}q)+(\mathrm{S}r+\mathrm{V}r)$$
$$=\mathrm{S}(p+q+r)+\mathrm{V}(p+q+r) \quad \text{(v.)}$$

contains the principle of the centre of mass. It asserts that the point $p+q+r$ is situated at the centre of mass of p, q and r, and that its weight $\mathrm{S}(p+q+r)$ is the sum of the weights of the three points. In another form,

$$m_1(1+\alpha_1)+m_2(1+\alpha_2)+m_3\ 1+\alpha_3)$$
$$=(m_1+m_2+m_3)\left(1+\frac{m_1\alpha_1+m_2\alpha_2+m_3\alpha_3}{m_1+m_2+m_3}\right).$$

Ex. 1. The middle point of the line AB is $\frac{1}{2}(\mathrm{A}+\mathrm{B})$.

Ex. 2. Interpret the relation

$$(\mathrm{S}p+\mathrm{V}q)+(\mathrm{S}q+\mathrm{V}r)+(\mathrm{S}r+\mathrm{V}p)=p+q+r,$$

regarding $\mathrm{S}p+\mathrm{V}q$, etc., as representing weighted points.

Ex. 3. The centre of mass of equal and opposite weights is at infinity.

Ex. 4. The equations of the line a, b and of the plane a, b, c are

$$q=xa+yb, \quad q=xa+yb+zc,$$

where x, y and z are scalars.

Ex. 5. Corresponding points of similar divisions on the lines ab and cd are

$$\frac{a}{\mathrm{S}a}+t\frac{b}{\mathrm{S}b}, \quad \frac{c}{\mathrm{S}c}+t\frac{d}{\mathrm{S}d};$$

and corresponding points of homographic divisions on the same lines are

$$a+tb, \quad c+td,$$

t being a variable scalar.

[See Art. 37, p. 41.]

Ex. 6. The equation $q=a+2bt+ct^2$ represents a conic.

Ex. 7. The equation

$$q=a+tb+u(c+td)$$

represents a ruled quadric, t and u being variable scalars.

ART. **144.** In order to develop this method, it becomes necessary to employ certain special symbols, and with one exception these are to be found in Art. 365 of Hamilton's *Elements of Quaternions*, though in quite a different connection.

For any pair of points, we write

$$(a, b)=b\mathrm{S}a-a\mathrm{S}b, \quad [a, b]=\mathrm{V}\,.\,\mathrm{V}a\mathrm{V}b; \quad \text{(I.)}$$

and in particular, for points of unit weight ($\mathrm{A}=1+\alpha$, $\mathrm{B}=1+\beta$), these become

$$(\mathrm{A}, \mathrm{B})=\mathrm{B}-\mathrm{A}=\beta-\alpha, \quad [\mathrm{A}, \mathrm{B}]=\mathrm{V}\,.\,\mathrm{VAVB}=\mathrm{V}\alpha\beta=\mathrm{V}\alpha(\beta-\alpha). \quad \text{(II.)}$$

Thus (a, b) is the product of the weights into the vector connecting the points, and $[ab]$ is the product of the weights into the moment of the vector connecting the points with respect to the scalar point or origin. The two functions (a, b) and $[a, b]$ completely determine the line ab.

For any three points we write

$$\left.\begin{array}{l}[a, b, c]=(a, b, c)-[b, c]\mathrm{S}a-[c, a]\mathrm{S}b-[a, b]\mathrm{S}c, \\ (a, b, c)=\mathrm{S}[a, b, c]=\mathrm{S}\,.\,\mathrm{V}a\mathrm{V}b\mathrm{V}c=\mathrm{S}a[b, c],\end{array}\right\}\ldots\text{(III.)}$$

and for unit points $\mathrm{A}=1+\alpha$, $\mathrm{B}=1+\beta$, $\mathrm{C}=1+\gamma$, these become

$$[\mathrm{A}, \mathrm{B}, \mathrm{C}]=\mathrm{S}\alpha\beta\gamma-\mathrm{V}\beta\gamma-\mathrm{V}\gamma\alpha-\mathrm{V}\alpha\beta,\ (\mathrm{ABC})=\mathrm{S}\,.\,\alpha\beta\gamma.\ \ldots\text{(IV.)}$$

Hence it appears that the quaternion $[a, b, c]$ determines the plane of the points, and regarded as a point symbol $[a, b, c]$ represents the reciprocal of the plane with respect to the unit sphere having its centre at the scalar point. For the vector $\mathrm{V}[abc]:\mathrm{S}[abc]$ is minus the reciprocal of the vector perpendicular from the origin on the plane $\mathrm{S}\rho\mathrm{V}(\beta\gamma+\gamma\alpha+\alpha\beta)=\mathrm{S}\alpha\beta\gamma$; that is, its extremity terminates at the pole of the plane with respect to the unit sphere. The symbol (a, b, c) is the sextupled volume of the pyramid OABC multiplied by the weights $\mathrm{S}a\mathrm{S}b\mathrm{S}c$.

Any quaternion may therefore be regarded as representing at pleasure a plane or a point—reciprocals with respect to the unit sphere.

The last special symbol we require at present is

$$(abcd)=\mathrm{S}a[bcd]\,;\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(V.)}$$

or for unit points,

$$(\mathrm{ABCD})=\mathrm{S}\beta\gamma\delta-\mathrm{S}\alpha\gamma\delta+\mathrm{S}\alpha\beta\delta-\mathrm{S}\alpha\beta\gamma.\ \ldots\ldots\ldots\text{(VI.)}$$

Thus (ABCD) is the sextupled volume of the tetrahedron ABCD, and $(abcd)$ is the same volume multiplied by the product of the weights.

It will be observed that the five functions are *combinatorial*, that is to say, they remain unchanged when to any of the quaternions involved in one of the functions is added a sum of products of the other quaternions multiplied by scalar coefficients. For example, $[a+xb+yc, b, c]=[a, b, c]$. More generally when the constituent quaternions are replaced by linear functions of themselves with scalar multipliers, the functions are merely multiplied by a scalar. If any linear relation with scalar coefficients connects the constituents of a function, the value of the function is zero. If any two constituents are transposed the function changes sign, and in fact the laws of combination of the rows or columns of an ordinary scalar determinant are obeyed by the constituents of the functions.

ART. **145.** In terms of these functions, the equation of the line ab and of the plane abc are respectively

$$[q, a, b]=0, \quad (q, a, b, c)=0; \qquad \text{(I.)}$$

the first expressing that q, a and b are linearly connected, or that the plane qab is indeterminate; the second requiring the volume (QABC) to be zero.

The equation of the line ab may also be written in the form

$$(pqab)=0, \qquad \text{(II.)}$$

where p is a point wholly arbitrary; and the equation of the plane may be replaced by

$$Sql=0, \quad \text{where} \quad l=[abc], \qquad \text{(III.)}$$

the point l being, as we have said, the reciprocal of the plane with respect to the unit sphere*

$$S \, . \, q^2=0, \qquad \text{(IV.)}$$

or $S \, . \, (1+OQ)^2=0$, or $OQ^2+1=0$. Putting $L=1+OL$, the equation of the plane takes the known vector form $S(1+OQ)(1+OL)=0$

The plane at infinity is

$$Sq=0, \qquad \text{(V.)}$$

this being the reciprocal of the scalar point (the centre) with respect to the unit sphere; or otherwise if q represents a point at infinity it is a vector (Art. 143, p. 263), so that $Sq=0$.

The formulae of reciprocation

$$([abc]; [abd])=[ab](abcd); \quad [[abc]; [abd]]=-(ab)(abcd), \quad \text{(VI.)}$$

are worthy of notice. They connect two points a and b with two points $[abc]$ and $[abd]$ on the reciprocal of the line ab, and are easily verified by vectors. Formulae, such as these, are often suggested by the forms of the expressions. For example, the left-hand members of the above relations evidently vanish if a, b, c and d are linearly connected. We infer that $(abcd)$ is a factor, and the remaining factor must be a combination of (ab) and $[ab]$.

It is often useful to observe that if i, j and k are mutually rectangular unit vectors,

$$(1, i)=i, \quad [i, j]=k, \quad [1, i, j]=-k,$$
$$[i, j, k]=-1, \quad (1, i, j, k)=-1; \qquad \text{(VII.)}$$

and relations such as these may be employed to ascertain the numerical factors in expressions such as (VI.).

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* In ordinary homogeneous coordinates the auxiliary quadric is generally taken to be $x^2+y^2+z^2+w^2=0$. It is more convenient in quaternions to employ the unit sphere as the auxiliary. There is however no loss of generality. (Compare Art. 153 (x.), p. 284.)

Ex. 1. Two lines, a, b, and c, d, intersect if

$$(abcd)=0.$$

(a) This condition may be also written in the form

$$S(ab)[cd]+S[ab](cd)=0.$$

Ex. 2. The point of intersection of three planes

$$Slq=0,\ Smq=0,\ Snq=0 \text{ is } q=[l,\ m,\ n].$$

Ex. 3. The line of intersection of two planes $Slq=0$, $Smq=0$ is

$$q=[l,\ m,\ n],$$

where n is an arbitrary quaternion.

Ex. 4. If four planes l, m, n, p have a common point

$$(l,\ m,\ n,\ p)=0.$$

Ex. 5. The line a, b intersects the plane $Slq=0$ in the point

$$aSlb-bSla.$$

Ex. 6. The general equation of a conic is

$$q=at^2+2bt+c,$$

where t is a scalar parameter.

(a) The expression $\quad q=at_1t_2+b(t_1+t_2)+c$

represents the pole of the chord joining the points t_1 and t_2, or the tangent at t_1 if t_2 is variable.

(b) The pole of the line in which the plane $Slq=0$ meets that of the conic is

$$q=aSlc-2bSlb+cSla.$$

(c) The centre is $\quad q=aSc-2bSb+cSa.$

(d) The conic is a parabola if $SaSc=(Sb)^2$.

(e) What kind of a conic is represented by

$$q=\mathrm{A}t^2+2\mathrm{B}t+\mathrm{C}\,?$$

(f) If q, q_1, q_2, q_3 and q_4 are any five points on a conic, and if t, t_1, t_2, t_3 and t_4 are the corresponding parameters, the anharmonic of the pencil $q\cdot\{q_1q_2q_3q_4\}$ is

$$\frac{(q-q_1,\ q-q_2)\cdot(q-q_3,\ q-q_4)}{(q-q_2,\ q-q_3)\cdot(q-q_4,\ q-q_1)}=\frac{(t_1-t_2)(t_3-t_4)}{(t_2-t_3)(t_4-t_1)}.$$

Ex. 7. The general twisted cubic is

$$q=(a,\ b,\ c,\ d\between t1)^3.$$

(a) The equation $\quad q=(a,\ b,\ c,\ d\between t_1,\ 1\between t_2,\ 1)^2$

represents the tangent at the point t_2, t_1 being variable.

(b) The osculating plane at a point is

$$q=(a,\ b,\ c,\ d\between t_1,\ 1\between t_2,\ 1\between t_3,\ 1),$$

two of the scalars t_1, t_2, t_3 being variable and the other being fixed.

(c) The equation in (a) represents the tangent line developable when t_1 and t_2 both vary.

(d) If t_1 is given it represents the conic in which the osculating plane at t_1 cuts the developable.

(e) The locus of the poles of a fixed plane $Slq=0$ with respect to these conics is the conic,

$$q=t_1^2(aScl-2bSbl+cSal)+t_1(aSdl-bScl-cSbl+dSal)+bSdl-2cScl+dSbl.$$

(*f*) The osculating planes at the points in which the plane $Slq=0$ meets the curve intersect in the point

$$q = a\mathrm{S}dl - 3b\mathrm{S}cl + 3c\mathrm{S}bl - d\mathrm{S}al,$$

and this point lies in the plane.

(*g*) The symbol of the osculating plane $Spq=0$ at the point t is

$$p = [at+b,\ bt+c,\ ct+d];$$

and this equation also represents the cuspidal edge of the reciprocal developable.

(*h*) The last equation may be written in the form

$$p = t^3[abc] + t^2[abd] + t[acd] + [bcd].$$

(*i*) The symbol of the plane containing three points t_1, t_2, t_3 is

$$p = 3t_1t_2t_3[abc] + \Sigma t_2t_3 \,.\, [abd] + \Sigma t_1 \,.\, [acd] + 3[bcd].$$

(*j*) The anharmonic of the group of planes joining two variable points on the cubic to four fixed points is constant.

ART. **146**. Hamilton has given two relations connecting five arbitrary quaternions,

$$a(bcde) + b(cdea) + c(deab) + d(eabc) + e(abcd) = 0 \ \ldots\ldots(\text{I.})$$

and $$e(abcd) = [bcd]\mathrm{S}ae - [acd]\mathrm{S}be + [abd]\mathrm{S}ce - [abc]\mathrm{S}de;\ \ldots(\text{II.})$$

which are of great importance and which correspond to the vector relations

$$\delta\mathrm{S}\alpha\beta\gamma = \alpha\mathrm{S}\beta\gamma\delta + \beta\mathrm{S}\gamma\alpha\delta + \gamma\mathrm{S}\alpha\beta\delta = \mathrm{V}\beta\gamma\mathrm{S}\alpha\delta + \mathrm{V}\gamma\alpha\mathrm{S}\beta\delta + \mathrm{V}\alpha\beta\mathrm{S}\gamma\delta.$$

The first has been virtually proved in Art. 39, p. 43, and we may at once verify it by writing

$$xa + yb + zc + wd + ve = 0,$$

where x, y, z, w and v are scalars to be determined. From this, by the combinatorial property, we have

$$0 = (a,\ b,\ c,\ xa + yb + zc + wd + ve) = (a,\ b,\ c,\ wd + ve),$$

which gives the ratio of w to v. This relation enables us to express any point in terms of four given points, so that we may if we choose use an arbitrary tetrahedron of reference, for example $abcd$.

The second shows how to refer any point to four given planes

$$\mathrm{S}aq = 0, \quad \mathrm{S}bq = 0, \quad \mathrm{S}cq = 0, \quad \mathrm{S}dq = 0;$$

and the truth of the formula may be verified by observing that we get consistent results when we operate with Sa, Sb, Sc and Sd.

It will be observed that the relations (I.) and (II.) are linear with respect to each of the five quaternions, so that the weights of the points do not enter. In fact, just as in tetrahedral coordinates, geometrical relations depend on homogeneous functions of the quaternions. Though it is in general distinctly disadvantageous to employ any system of coordinates in quaternion investigations, or even to refer in thought to any tetrahedron or axes of reference until a problem has been reduced to its ultimate simplicity, yet it is worth while observing that if we express a variable quaternion q in terms of four given quaternions a, b, c, d by means of the relation

$$q = xa + yb + zc + wd, \qquad \text{(III.)}$$

the scalars x, y, z and w are the anharmonic coordinates of Art. 40, p. 43.

Ex. 1. The line de meets the plane abc in the point

$$d(abce) - e(abcd).$$

Ex. 2. Show that

$$([abc],\ [def]) = [ef](abcd) + [fd](abce) + [de](abcf),$$
$$[[abc],\ [def]] = -(ef)(abcd) - (fd)(abce) - (de)(abcf).$$

[Compare Art. 145 (VI.). Four points on the line of intersection of the planes abc and def are $d(abce) - e(abcd)$ and $d(abcf) - f(abcd)$, and the functions $[a'b']$ and $-(a'b')$ for two points on the line are proportional to the right-hand members of the above. The weights are correct, and it only remains to determine the numerical factors. Putting $d = a$ and $e = b$, we verify the signs by the equations cited.]

Ex. 3. The point of intersection of the planes abc, def and ghi is

$$[[abc],\ [def],\ [ghi]] = \begin{vmatrix} a & b & c \\ (adef) & (bdef) & (cdef) \\ (aghi) & (bghi) & (cghi) \end{vmatrix}.$$

[Equating the left-hand member to $xa + yb + zc$, we have

$$x(adef) + y(bdef) + z(cdef) = 0, \text{ etc.},$$

and to determine the factor we may put

$$a = 1,\ b = i,\ c = j,\ [abc] = -k,\ [def] = i,\ [ghi] = j.$$

The left-hand member becomes $+1$, and the determinant also reduces to $+1$.]

Ex. 4. Given four triangles $a_nb_nc_n$, where $n = 1$, 2, 3 or 4, show that six times the volume of the tetrahedron determined by their planes is

$$\frac{1}{\Pi(a_nb_nc_n)} \cdot \begin{vmatrix} (a_1a_2b_2c_2) & (b_1a_2b_2c_2) & (c_1a_2b_2c_2) \\ (a_1a_3b_3c_3) & (b_1a_3b_3c_3) & (c_1a_3b_3c_3) \\ (a_1a_4b_4c_4) & (b_1a_4b_4c_4) & (c_1a_4b_4c_4) \end{vmatrix}.$$

[This follows from the last example.]

Ex. 5. Establish the identities

$$\begin{vmatrix} Saa' & Sab' & Sac' \\ Sba' & Sbb' & Sbc' \\ Sca' & Scb' & Scc' \end{vmatrix} = -S[abc][a'b'c'];$$

$$\begin{vmatrix} Saa' & Sab' & Sac' & Sad' \\ Sba' & Sbb' & Sbc' & Sbd' \\ Sca' & Scb' & Scc' & Scd' \\ Sda' & Sdb' & Sdc' & Sdd' \end{vmatrix} = -(abcd)(a'b'c'd').$$

[The first determinant is combinatorial in a, b and c and also in a', b' and c'. It vanishes if either triangle reduces to a line, and conversely. Hence it must be a scalar function of $[abc]$ and of $[a'b'c']$, that is (having regard to the weights) it must be of the form

$$x\mathrm{SV}[abc]\mathrm{V}[a'b'c]+y\mathrm{S}[abc]\mathrm{S}[a'b'c'],$$

where x and y are numerical factors. For $a=a'=i$, $b=b'=j$, $c=c'=k$ we get $y=-1$, and for $a=a'=1$, $b=b'=i$, $c=c'=j$ we find $x=-1$.]

Ex. 6. Prove that

$$\begin{vmatrix} \mathrm{S}aa' & \mathrm{S}ab' \\ \mathrm{S}ba' & \mathrm{S}bb' \end{vmatrix} = \mathrm{S}(ab)(a'b') - \mathrm{S}[ab][a'b'].$$

[This is most easily proved by vectors. Compare Art. 145, Ex. 1.]

Ex. 7. Find the equation of the hyperboloid having three given generators ab, $a'b'$ and $a''b''$.

[There are various methods of finding this equation, but we shall give a method to illustrate the use of Ex. 3. If p and q are any two points on a generator of the opposite system to the given lines, the conditions of intersection are $(pqab)=0$, $(pqa'b')=0$, $(pqa''b'')=0$. Regarding these conditions as the equations of planes, p being the variable point, the condition that the planes should intersect in a line is $[[qab][qa'b'][qa''b'']]=0$, which becomes $(aqa'b')(bqa''b'')-(bqa'b')(aqa''b'')=0$.]

ART. **147**. The results of the last article are particular cases of a very general theory applicable not only to quaternions but to any operators or quantities which are associative and commutative in addition.*

If $f(a, b)$ is a function of two quaternions distributive with respect to each, the function

$$f(a, b)-f(b, a) \qquad \text{(I.)}$$

is combinatorial in a and b, for it remains unchanged when we replace a by $a+yb$ or b by $b+xa$, because

$$f(a+yb, b)=f(a, b)+yf(b, b) \text{ and } f(b, a+yb)=f(b, a)+yf(b, b).$$

In like manner if $f(a, b, c)$ is distributive with respect to a, b and c the function $f(a, b, c)-f(b, a, c)$ is combinatorial in a and b; the function formed by subtracting from this the result of interchanging a and c is combinatorial in a and b and also in a and c; and the function of six terms

$$\Sigma \pm f(a, b, c) \qquad \text{(II.)}$$

formed by transposing a, b and c in $f(a, b, c)$ in every possible way, by changing the sign after every transposition of a pair of constituents and by adding the results together, is combinatorial in a, b and c. Similarly if $f(a, b, c, d)$ is distributive in a, b, c and d, the sum

$$\Sigma \pm f(a, b, c, d) \qquad \text{(III.)}$$

*See an interesting paper by Prof. A. S. Hathaway, *Proc. Acad. of Science*, 1897.

is combinatorial in a, b, c and d; and finally

$$\Sigma \pm f(a, b, c, d, e) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(IV.)}$$

is combinatorial in a, b, c, d, e and vanishes identically because the five quaternions are linearly connected.

It is geometrically evident from Art. 144, that every combinatorial function of two quaternions a and b must be a function of (ab) and $[ab]$—the two vectors which determine the line ab. Every combinatorial function of a, b and c must be a function of $[abc]$ which determines the plane abc; and the only combinatorial function of four points is $(abcd)$—the sextupled volume of the tetrahedron determined by them. Hence (II.) is a linear function of $[abc]$ and (III.) is the product of a quaternion by the scalar $(abcd)$.

Now in forming these sums, we may proceed step by step. For example, let us transpose $bcde$ in $f(a, b, c, d, e)$, leaving a unchanged. We obtain the sum

$$\Sigma \pm f(a_0, b, c, d, e),$$

where the temporary suffix applied to a denotes that it is free from the operation indicated by $\Sigma\pm$. Next interchange a and b and change the sign and permute a, c, d, e, leaving b unchanged. We get

$$-\Sigma \pm f(b_0, a, c, d, e).$$

Finally the vanishing combinatorial function (IV.) is expanded in the form

$$\Sigma \pm f(a_0 bcde) - \Sigma \pm f(b_0 acde) + \Sigma \pm f(c_0 abde) - \Sigma \pm f(d_0 abce) + \Sigma \pm f(e_0 abcd) = 0, \quad \text{(V.)}$$

and this general result includes Art. 146 (I.) as a particular case.

Again we may leave two or more quaternions fixed and add together the sums obtained, so that for example

$$\Sigma \pm f(a_0 b_0 cd) - \Sigma \pm f(a_0 c_0 bd) + \text{etc.} = \Sigma \pm f(abcd). \quad\ldots\ldots\text{(VI.)}$$

These expansions correspond to the expansions of determinants by minors.

Ex. Find the *sources* of the functions

$$([abc], d), \quad [[abc], d],$$

which are combinatorial in a, b and c, or in other words find linear functions of a, b, c from which the combinatorial functions may be derived by summation and transposition.

[Since $\quad (abc)\,.\,\mathrm{V}d = [bc]\mathrm{S}\,.\,a\mathrm{V}d + [ca]\mathrm{S}\,.\,b\mathrm{V}d + [ab]\mathrm{S}\,.\,c\mathrm{V}d$

and $\quad \mathrm{V}[abc]\mathrm{S}d = -[bc]\mathrm{S}a\mathrm{S}d - [ca]\mathrm{S}b\mathrm{S}d - [ab]\mathrm{S}c\mathrm{S}d,$

the first expression is $\Sigma \pm bc\mathrm{S}ad_0$. Similarly the second expression is

$$-\mathrm{V}\,.\,[bc]\mathrm{V}d\,.\,\mathrm{S}a - \mathrm{V}\,.\,[ca]\mathrm{V}d\mathrm{S}b - \mathrm{V}\,.\,[ab]\mathrm{V}d\mathrm{S}c,$$

and the function may be derived from $-\mathrm{V}b\mathrm{S}\mathrm{V}c\mathrm{V}d_0\,.\,\mathrm{S}a$ or from $-b\mathrm{S}cd_0\mathrm{S}a$, certain parts of this latter expression vanishing under transposition and summation. As a determinant, the function is

$$[[abc]d]=\begin{vmatrix} a & b & c \\ Sa & Sb & Sc \\ Sad & Sbd & Scd \end{vmatrix},$$

and this may be deduced directly as follows. We may assume

$$[[abc]d]=xa+yb+zc \text{ since } S[abc][[abc]d]=0\,;$$

and we have $\qquad xSa+ySb+zSc=0,\ xSad+ySbd+zScd=0.$

The numerical factor of the determinant resulting from this may be determined by substituting special values for a, b, c, d.]

ART. **148**. We shall now consider the general linear transformation of points in space.

In analogy with the linear vector function, the linear quaternion function fq is a function which satisfies

$$f(a+b)=fa+fb \quad \text{.........................(I.)}$$

for all pairs of quaternions a and b.

The relation $\qquad p=fq \quad$ ....(II.)

represents the general linear transformation from points q to points p, lines and planes

$$q=a+tb,\ q=a+tb+uc,$$

becoming lines and planes

$$p=fa+tfb,\ q=fa+tfb+ufc,$$

and anharmonic properties being preserved.

If four given quaternions, a, b, c and d, are converted by a linear transformation into four others, a', b', c' and d', the function which effects this transformation is (compare Art. 62 (IV.), p. 88, and Art. 146 (I.)

$$fq=-\{a'(bcdq)+b'(cdqa)+c'(dqab)+d'(qabc)\}(abcd)^{-1};\ \text{(III.)}$$

and this function is in the quadrinomial form. To reduce a function to the quadrinomial form, we may arbitrarily assume any four quaternions a, b, c, d and use either of the relations connecting five quaternions. Taking the second,

$$fq=\{f[bcd]Saq-f[acd]Sbq+f[abd]Scq$$
$$-f[abc]Sdq\}(abcd)^{-1}, \text{.....(IV.)}$$

and thus a linear quaternion function depends on sixteen constants, four constants being involved in each of the four quaternions $f[bcd]$, etc.

In (III.) we supposed the weights given. Let us now determine a function which shall convert five given points A, B, C, D, E into five others A′, B′, C′, D′, E′, paying no attention to the weights. Such a function is

$$fq = \frac{A'(BCDq)(B'C'D'E')}{(BCDA)(BCDE)} + \frac{B'(ACDq)(A'C'D'E')}{(ACDB)(ACDE)}$$
$$+ \frac{C'(ABDq)(A'B'D'E')}{(ABDC)(ABDE)} + \frac{D'(ABCq)(A'B'C'E')}{(ABCD)(ABCE)}; \quad \ldots(\text{v.})$$

for replacing q by A we get $f\text{A} = \text{A}'(\text{B}'\text{C}'\text{D}'\text{E}')(\text{BCDE})^{-1}$, etc., and putting $q = \text{E}$, we have $f\text{E} = \text{E}'(\text{A}'\text{B}'\text{C}'\text{D}')(\text{ABCD})^{-1}$ in virtue of the relation connecting five quaternions. Thus the function (v.) effects the required transformation, and it is evidently determinate to a scalar factor. (Compare Art. 65, Ex. 5, p. 92.)

ART. **149**. *A linear function f being regarded as producing a transformation of points, the inverse of its conjugate f'^{-1} produces the corresponding tangential transformation.*

For any quaternions p and q,

$$Spq = Spf^{-1}q' = Sq'f'^{-1}p = Sq'p_{\prime} \text{ if } q' = fq,\ p_{\prime} = f'^{-1}p. \quad \ldots\ldots(\text{I.})$$

Hence any plane $Spq = 0$, in which q is the current point and p the symbol of the plane, becomes after the transformation $Sp_{\prime}q' = 0$, where q' is the transformed current point and where $p_{\prime}$ is the transformed symbol of the plane. In other words when points are transformed by the operation of f, planes are transformed by the operation of f'^{-1}.

ART. **150**. Now the symbol of the plane may be expressed in terms of three points in the plane (Art. 145, p. 266), and therefore for some scalar factor n,

$$nf'^{-1}[abc] = [fa, fb, fc] = F'[a, b, c], \quad \ldots\ldots\ldots\ldots(\text{I.})$$

since we may either transform the symbol of the plane in one step by f'^{-1} or we may transform the points a, b, c which enter into the symbol by f. The function F' is a new linear function analogous to Hamilton's ψ', and it is connected with f'^{-1} by the relation

$$n = f'F' = F'f'. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{II.})$$

The scalar n may be explicitly expressed in terms of four *arbitrary* points, a, b, c, d, by operating with $S\,.fd$ on (I.), when we find

$$n(abcd) = (fafbfcfd) = S[abc]Ffd, \quad \ldots\ldots\ldots\ldots(\text{III.})$$

where F is the conjugate of F'.

Thus in addition to (II.) we have,

$$n = fF = Ff; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{IV.})$$

and we may also write

$$\left.\begin{aligned} n(abcd) &= (fafbfcfd) = (f'af'bf'cf'd), \\ F[abc] &= [f'af'bf'c] = nf^{-1}[abc]. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(\text{V.})$$

Replacing f by $f+t$, where t is a scalar, the relations

$$n_t = n + tn' + t^2n'' + t^3n''' + t^4 = f_tF_t = (f+t)(F+tG+t^2H+t^3) \quad \text{(VI.)}$$

are obtained, where the new scalars n', n'', n''' and the new linear functions G and H are defined by

$$\left.\begin{aligned} n'(abcd) = \Sigma(afbfcfd);\quad n''(abcd) = \Sigma(abfcfd);\\ n'''(abcd) = \Sigma(abcfd);\\ G[abc] = [a, f'b, f'c] + [f'a, b, f'c] + [f'a, f'b, c];\\ H[abc] = [f'a, b, c] + [a, f'b, c] + [a, b, f'c]. \end{aligned}\right\} \quad \text{(VII.)}$$

Moreover, on account of the arbitrariness of t in (VI.),

$$n = fF,\ n' = fG + F,\ n'' = fH + G,\ n''' = f + H; \ldots\text{(VIII.)}$$

and from the symbolical equations may be deduced the following explicit expressions for the auxiliary functions

$$H = n''' - f;\ G = n'' - n'''f + f^2;\ F = n' - n''f + n'''f^2 - f^3; \quad \text{(IX.)}$$

and the symbolic quartic

$$n - n'f + n''f^2 - n'''f^3 + f^4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\text{(X.)}$$

satisfied by the function f.

ART. **151.** Let t_1, t_2, t_3 and t_4 be the roots of the scalar quartic

$$t^4 - n'''t^3 + n''t^2 - n't + n = 0, \ldots\ldots\ldots\ldots\ldots\ldots\text{(I.)}$$

so that the symbolic quartic may be expressed in the form

$$(f-t_1)(f-t_2)(f-t_3)(f-t_4) = 0. \ldots\ldots\ldots\ldots\ldots\ldots\text{(II.)}$$

It follows just as in the case of the vector function that

$$(f-t_1)q_1 = 0, \quad \text{where} \quad (f-t_2)(f-t_3)(f-t_4)q = q_1, \ldots\ldots\text{(III.)}$$

and that q_1 is a fixed point—a united point of the transformation—one of four q_1, q_2, q_3 and q_4. The point q is quite arbitrary.

The equations

$$p = (f-t_1)q, \quad p = (f-t_1)(f-t_2)q, \ldots\ldots\ldots\ldots\ldots\text{(IV.)}$$

represent respectively a united plane of the transformation and a united line—the plane $[q_2, q_3, q_4]$ and the line q_3q_4.

We have also by the property of the conjugate,

$$Sq_1'p = Sq_1'(f-t_1)q = 0 \quad \text{if} \quad (f'-t_1)q_1' = 0; \ldots\ldots\ldots\text{(V.)}$$

and thus the united points (q_1', q_2', q_3' and q_4') of the conjugate (f') are the reciprocals with respect to the unit sphere (Art. 145) of the united planes of f. In other words, *the united points of a function and of its conjugate form tetrahedra reciprocal with respect to the unit sphere.*

Ex. 1. Prove that fq may be reduced to the form

$$fq = (e+\epsilon)Sq + S\epsilon'Vq + \phi Vq,$$

and determine its latent quartic in terms of the linear vector function ϕ, the vectors ϵ and ϵ' and the scalar e.

[By the distributive principle $fq = f\mathrm{S}q + f\mathrm{V}q$, etc. To determine the quartic assume $fq = tq = t(\mathrm{S}q + \mathrm{V}q)$, and equate scalar and vector parts. We find $(e-t)\mathrm{S}q + \mathrm{S}\epsilon'\mathrm{V}q = 0$, $(\phi - t)\mathrm{V}q + \epsilon\mathrm{S}q = 0$, so that

$$(e-t) - \mathrm{S}\epsilon'(\phi - t)^{-1}\epsilon = 0.]$$

Ex. 2. Construct a function with four zero latent roots.

[Assume $fa = b$, $fb = c$, $fc = d$, $fd = 0$.]

Ex. 3. Examine the nature of the symbolic equation satisfied by the function

$$fq = a(bcdq) + b(cdqa) + c'(dqab) + d'(qabc).$$

[Every point $a + ub$ on the line a, b, is a united point of the function, and the F function of $fq - (abcd)q$ vanishes identically. The quartic degrades into a cubic.]

Ex. 4. Construct a function satisfying a symbolic quadratic.

[This may arise from one of two causes. The function may have two line loci of united points a, b and c, d; or it may have a plane locus of united points a, b, c. In the first case the latent quartic is a perfect square. In the second it has a triple root. For full details on these matters see *Phil. Trans.*, vol. 201, viii.]

Ex. 5. Prove that two real lines remain unaltered by the general real linear transformation.

[If the roots are all real of course the six edges of the united tetrahedron remain unaltered. If the roots are all imaginary, they occur in conjugate pairs, and the united points must be of the form $a \pm \sqrt{-1}b$, $c \pm \sqrt{-1}d$. The lines ab and cd are real and remain unchanged.]

ART. **152**. Just as in the case of the vector function, we obtain two new functions

$$f_0 = \tfrac{1}{2}(f + f'), \quad f_{,} = \tfrac{1}{2}(f - f'), \quad \text{(I.)}$$

on combining a function and its conjugate by addition and subtraction.

The function f_0 is self-conjugate and the function $f_{,}$ is the negative of its conjugate, or

$$f_0 = f_0', \quad f_{,} = -f_{,}', \quad \text{(II.)}$$

as we see at once by the property of the conjugate.

Since fq is the general linear function of q, $\mathrm{S}qfq$ or $\mathrm{S}qf_0q$ is the general scalar quadratic function, and

$$\mathrm{S}qf_0q = 0 \quad \text{(III.)}$$

represents the general quadric surface, the surface being quite arbitrary both in shape and position, and not now referred to its centre as in Art. 72, p. 106.

In like manner $\qquad \mathrm{S}pf_{,}q = 0 \quad \text{(IV.)}$

is the general equation of a linear complex, or of a family of lines p, q satisfying a single condition of the first order. For if we replace p by $p + tq$ the equation remains unchanged, for we have generally, by the property of the conjugate (II.),

$$\mathrm{S}qf_{,}q = -\mathrm{S}qf_{,}q = 0.$$

The equations $\quad Sqf_0a=0, \quad Sqf_{,}b=0 \quad$ ....(v.)

represent respectively the polar plane of the point a with respect to the quadric, and the plane containing the lines of the complex which pass through b. The first equation may be deduced from the result of substituting $a+tq$ in the equation of the quadric, when we find

$$Saf_0a+2tSqf_0a+t^2Sqf_0q=0,$$

and if q is on the polar plane, the points in which the line aq meets the quadric must be expressible by $a+tq$, $a-tq$, because the polar plane is the locus of harmonic means, and the points a, $a+tq$, q, $a-tq$ form a harmonic range.

If $Slq=0$ is an arbitrary plane we see on comparison with (v.) that the pole of the plane with respect to the quadric is $f_0^{-1}l$, and that the point of concourse of the lines of the complex which lie in the plane is $f_{,}^{-1}l$. It also appears that

$$Slf_0^{-1}l=0 \quad \text{and} \quad Smf_{,}^{-1}l=0 \quad \text{..................(vi.)}$$

represent respectively the tangential equation of the quadric, or the equation of the reciprocal quadric; and the tangential equation of the complex (the intersection of the planes $Slq=0$, $Smq=0$ being a line of the complex), or the equation of the reciprocal complex.

A complete account of the nature of the united points of the functions f_0 and $f_{,}$ is furnished by the theorem of Art. 151. Since f_0 is its own conjugate, each of its united points is reciprocal to the plane containing the remaining three, or the tetrahedron of united points is self-conjugate to the sphere of reciprocation. We saw in Art. 67, p. 96, that it is impossible for a real self-conjugate linear vector function to have a pair of equal roots without having indeterminate axes, and this because a real line cannot be perpendicular to itself. But a real self-conjugate linear quaternion function may have two of its united points coalesced into a single point provided the point is on the sphere of reciprocation. The argument about real roots does not now apply. For suppose $a+\sqrt{-1}b$ and $a-\sqrt{-1}b$ to be two united points of a self-conjugate quaternion function, the condition of reciprocity is

$$S(a+\sqrt{-1}b)(a-\sqrt{-1}b)=Sa^2+Sb^2=0,$$

and this condition can be satisfied for real points a and b if one point (a) is inside and the other (b) is outside the sphere of reciprocation $Sq^2=0$.

As regards the function $f_{,}$, the most general form its symbolic quartic can have is

$$f_{,}^4+n_{,}''f_{,}^2+n_{,}=0 \quad \text{or} \quad (f_{,}^2-s)(f_{,}^2-s')=0, \quad \text{.........(vii.)}$$

because the same quartic is satisfied by the function and by its conjugate $(-f_{,})$. Supposing the united points to be a, a', b and b', where

$$f_{,}a=\sqrt{s}a,\ f_{,}b=-\sqrt{s}b,\ f_{,}a'=\sqrt{s'}a',\ f_{,}b'=-\sqrt{s'}b',$$

it is evident that a is the united point of the conjugate which corresponds to the root $-\sqrt{s}$, etc., and therefore by the theorem of Art. 151 we must have

$$Sa^2=0,\ Saa'=0,\ Sab'=0,\ Sb^2=0,\ Sba'=0,\ Sbb'=0,$$
$$Sa'^2=0,\ Sb'^2=0.$$

In other words the lines aa', ab', $a'b$ and bb' are generators of the unit sphere, or $aa'bb'$ is a quadrilateral on the sphere. The four lines are consequently all imaginary. By Ex. 5 of the last article it appears that the lines ab and $a'b'$ must be real; and since these lines are reciprocal to the unit sphere, one of them (ab) meets the sphere in two real points (a and b) and the other meets it in two imaginary points (a' and b'). Consequently one of the scalars (s) is positive and the other (s') is negative.

The common self-conjugate tetrahedron of two quadrics $Sqf_1q=0$, $Sqf_2q=0$ *has the united points of* $f_2^{-1}f_1$ *for its vertices.* For if $Slq=0$ is the polar of a point a for both quadrics

$$f_1a=t_1f_2a=l \quad\text{or}\quad f_2^{-1}f_1a=t_1a,\ \ldots\ldots\ldots\ldots\ldots(\text{VIII.})$$

so that a is a united point and t_1 the corresponding latent root of $f_1^{-1}f_2$. If b is a second united point corresponding to the root t_2,

$$Sbf_1a=t_1Sbf_2a=Saf_1b=t_2Saf_2b=0,$$

because the functions are self-conjugate. These relations are, however, geometrical consequences of (VIII.) and analogous expressions.

A little care is necessary when dealing with the equations of quadrics such as

$$Sq\,.\frac{f_1+xf_2}{f_1+yf_2}q=0 \quad\text{or}\quad Sq(f_1+xf_2)(f_1+yf_2)^{-1}q=0\,;$$

the second form of the equation shows that the function involved is not self-conjugate, although f_1 and f_2 are self-conjugate, unless f_1 is commutative with f_2.

Ex. 1. In terms of vectors prove that the forms of f and $f_{,}$ are

$$f_0(1+\rho)=e+\epsilon_0+S\epsilon_0\rho+\phi_0\rho\,;\ f_{,}(1+\rho)=\epsilon_{,}-S\epsilon_{,}\rho+V\eta\rho\,;$$

e being a scalar, ϵ_0, $\epsilon_{,}$, η being vectors and ϕ_0 being a self-conjugate linear vector function.

Ex. 2. Prove that the latent quartic of the function $f_{,}$ is

$$t^4+t^2(\epsilon_{,}^2-\eta^2)-(S\eta\epsilon_{,})^2=0$$

and verify the conclusions respecting the roots and united points of $f_{,}$.

Ex. 3. Prove that $$\mathrm{S}qf_2f_1^{-1}f_2q=0$$
is the locus of the poles of tangent planes of the quadric $\mathrm{S}qf_1q=0$ with respect to the quadric $\mathrm{S}qf_2q=0$.

Ex. 4. The locus of the points of concourse of lines of the complex $\mathrm{S}pf_,q=0$ which lie in the tangent planes of the quadric $\mathrm{S}qf_0q=0$ is the quadric $\mathrm{S}qf_,f_0^{-1}f_,q=0$.

Ex. 5. An arbitrary quadric and an arbitrary linear complex have a common quadrilateral of generators.

[This follows by expressing that the point of contact of a plane $\mathrm{S}lq=0$ with the quadric $\mathrm{S}qf_0q=0$ is the same as the point of concourse of the lines of the linear complex $\mathrm{S}pf_,q=0$ in the plane. We have $f_,a=tf_0a=ul$, where t and u are scalars, so that $f_0^{-1}f_,a=ta$. There are thus four points (a) through which pairs of the common generators pass, and these points are the united points of $f_0^{-1}f_,$.]

Ex. 6. If f_1 and f_2 are any two functions, prove that the latent quartics of f_1f_2 and of f_2f_1 are identical.

(a) Show also that the latent quartic of $f_0^{-1}f_,$ is of the form
$$t^4+t^2N''+N=0.$$

[The first part follows exactly as in the case of vector functions (Art. 71); the second is obtained by combining this principle with the fact that $-f_,f_0^{-1}$ is the conjugate of $f_0^{-1}f_,$.]

Ex. 7. If a, b, a' and b' are the united points of the function $f_0^{-1}f_,$ corresponding to the latent roots $+t$, $-t$, $+t'$ $-t'$, prove that if we take
$$q=\frac{xa+yb}{\sqrt{\mathrm{S}af_0b}}+\frac{za'+wb'}{\sqrt{\mathrm{S}a'f_0b'}},\quad p=\frac{x'a+y'b}{\sqrt{\mathrm{S}af_0b}}+\frac{z'a'+w'b'}{\sqrt{\mathrm{S}a'f_0b'}},$$
the equations of the quadric and the linear complex take the canonical forms
$$\tfrac{1}{2}\mathrm{S}qf_0q=xy+zw=0,\quad \mathrm{S}pf_,q=t(xy'-x'y)+t'(zw'-z'w).$$

Ex. 8. Prove that in any linear transformation the locus of a point which with its derived is in perspective with a fixed point is a twisted cubic.

[If a is a fixed point, the condition requires $[fq,\ q,\ a]=0$, so that q, fq and a are in a line. This equation may be replaced by $(f+t)q=ua$, or $q=u(f+t)^{-1}a$; and this curve meets an arbitrary plane $\mathrm{S}lq=0$ in the three points determined by the cubic $\mathrm{S}l(f+t)^{-1}a=0$, or $\mathrm{S}l(F+tG+t^2H+t^3)a=0$.]

Ex. 9. Prove that $$(q,fq,p,fp)=0$$
represents the quadratic complex of lines connecting points and their correspondents in the linear transformation produced by f.

(a) Prove that the reciprocal of this complex is the complex of the conjugate f',
$$(q,f'q,p,f'p)=0.$$

[If p and q are any two points on a line joining a point to its correspondent, we have for some scalars x, y, z, w, the relation $xp+yq=f(zp+wq)$. The complex follows on the elimination of the scalars.

If $\mathrm{S}lq=0$ and $\mathrm{S}mq=0$ are any two planes through q and its correspondent fq, we have $\mathrm{S}f'lq=0$, $\mathrm{S}f'mq=0$, and for some scalars $xl+ym=f'(zl+wm)$.]

Ex. 10. The lines joining points to their correspondents which meet an arbitrary right line a, b generate a quadric
$$(q,fq,a,b)=0.$$

Ex. 11. An arbitrary quadric $Sqf_0q=0$ has eight generators which join points to their correspondents in an arbitrary linear transformation.

[If the line q, fq is a generator of the quadric, the point q is one of the eight intersections of the three surfaces

$$Sqf_0q=0,\ Sfqf_0q=0,\ Sqf'f_0fq=0.$$

We shall see that this is the extension of Hamilton's theory of the "umbilicar generatrices."]

Ex. 12. The generalized normal at a point on a surface being defined as the line joining the point to the reciprocal of the tangent plane, prove that the normals of the doubly infinite family of quadrics

$$Sq\,.\,\frac{f+x}{f+y}\,.\,q=0 \quad (f=f')$$

compose the quadratic complex $(qfqpfp)=0$.

Ex. 13. The feet of the generalized normals of the doubly infinite family which pass through a given point a are given by

$$q=\frac{f+y}{f+z}\,.\,a,$$

where y and z are scalar parameters.

[Any point on the normal to the quadric x, y at the point q may be written in the form

$$\frac{f+z}{f+y}q=u\frac{f+x}{f+y}q+tq, \text{ where } u+t=1,\ ux+ty=z.]$$

Ex. 14. The locus of the feet of normals of the family of quadrics $y=const.$ which pass through a given point is a twisted cubic.

Ex. 15. A quadric has eight generators which are also normals.

[Expressing that $q=fa+xa$ is a generator of the quadric $Sqfq=0$, we have $Safa=0$, $Saf^2a=0$, $Saf^3a=0$, which give eight points a and eight corresponding normals. See Ex. 11.]

Ex. 16. Find the locus of poles of a fixed plane with respect to the system of quadrics

$$Sq\,.\,\frac{f+x}{f+y}\,.\,q=0.$$

(*a*) Prove that the plane $Slq=0$ touches one quadric if x is fixed, three if y is fixed, and that if no restriction is placed on x or y, the locus of the points of contact is a conic section.

[Compare generally Exs. 12, 13, 14. In general, if p is a point of contact, $p=\frac{f+y}{f+x}l$ with the condition $Sl\,.\,\frac{f+y}{f+x}l=0$, or

$$p=l+(y-x)(f+x)^{-1}l, \quad\text{or}\quad p=lSl(f+x)^{-1}l-(f+x)^{-1}lS\,.\,l^2$$

(since we need not attend to the weight of p). This reduces to a quadratic in x,

$$p=lSlFl-FlS\,.\,l^2+x(lSlGl-GlS\,.\,l^2)+x^2(lSlHl-HlS\,.\,l^2),$$

and the locus of p is a conic.]

Ex. 17. The tetrahedron formed by a point and by the poles of the tangent planes at the point to the three quadrics of a system inscribed in a developable taken with respect to any fourth quadric of the system, is self-conjugate with respect to this fourth quadric.

[The equation of a system of quadrics inscribed in a developable is $Sq(f_1+xf_2)^{-1}q=0$, this being the reciprocal of a system passing through a common curve. If x, y, z are the parameters of the quadrics which pass through a point p, and if $Sqf_2^{-1}q=0$ is the fourth quadric, the poles of the tangent planes are $f_2(f_1+xf_2)^{-1}p$, $f_2(f_1+yf_2)^{-1}p$, $f_2(f_1+zf_2)^{-1}p$. But

$$\begin{aligned} Sf_2(f_1+xf_2)^{-1}p\cdot f_2^{-1}\cdot f_2(f_1+yf_2)^{-1}p & \\ &= Sp(f_1+xf_2)^{-1}f_2(f_1+yf_2)^{-1}p \\ &= (x-y)^{-1}Sp(f_1+xf_2)^{-1}[(f_1+xf_2)-(f_1+yf_2)](f_1+yf_2)^{-1}p, \end{aligned}$$

and this vanishes since p lies on the three quadrics x, y, z. This in particular gives the theorem that confocal quadrics cut at right angles.]

Ex. 18. The locus of the poles of a plane $Sqa=0$ to the same system of quadrics is the line

$$q=(f_1+xf_2)a \quad\text{or}\quad [qf_1af_2a]=0\,;$$

the locus of the poles of the system of planes $Sq(a+tb)=0$ is the ruled quadric

$$q=(f_1+xf_2)(a+tb) \quad\text{or}\quad (qf_1af_1bf_2a)(qf_1bf_2af_2b)=(qf_1af_1bf_2b)(qf_1af_2af_2b)\,;$$

and the locus of the points of contact of the system of planes is the twisted cubic,
$$q=f_1(a+tb)S(a+tb)f_2(a+tb)-f_2(a+tb)S(a+tb)f_1(a+tb).$$

[In reducing the scalar equation of the quadric observe that the quaternion equation is of the form $q=a_1+xa_2+t(b_1+xb_2)$ and apply the identity Art. 146 (I.) to eliminate the arbitrary weight of q and the scalars x and t.]

Ex. 19. Prove that two planes can be drawn through an arbitrary line to be conjugate to every quadric of the system.

[If the planes $Sq(a+tb)=0$, $Sq(a+t'b)=0$ are conjugate to the quadrics $Sqf_1^{-1}q=0$ and $Sqf_2^{-1}q=0$, the conditions of conjugation

$$S(a+tb)f_1(a+t'b)=0, \quad S(a+tb)f_2(a+t'b)=0$$

lead on elimination of t or t' to a quadratic in t which determines the two planes in question. The case of exception arises when the line is a generator of some quadric. The two conditions become equivalent.]

Ex. 20. Examine the particular cases of the twisted cubic locus of Ex. 18.

[When the line of intersection of the planes is a generator of one of the quadrics, f_1 suppose, the locus becomes $q=f_1(a+tb)$. This shows that the points of contact are homographic with the tangent planes $Sq(a+tb)=0$. When the line of intersection of the planes is not a generator of some quadric, let $Sqa=0$ and $Sqb=0$ be the specially selected planes of the last example, and let $Sa(f_1+uf_2)a=0$, $Sb(f_1+vf_2)b=0$ so that u and v are the parameters of the quadrics touched by the two planes, then the equation of the cubic becomes

$$q=(f_1+uf_2)(a+tb)Saf_2a+(f_1+vf_2)(a+tb)t^2Sbf_2b.$$

The cubic is plane if $(f_1af_2af_1bf_2b)=0$. (See Ex. 9.)

The cubic degrades into a conic if $(f_1+vf_2)b=0$, or $(f_1+uf_2)a=0$, that is, if either of the planes is a united plane of $f_2^{-1}f_1$.]

Ex. 21. Determine the quadrics of the system $Sq(f_1+xf_2)^{-1}q=0$ touched by an arbitrary line.

[Taking the line to be the intersection of the planes $Sqa=0$, $Sqb=0$ of the last example, the condition of contact is most simply obtained by expressing that the reciprocal line $q=a+tb$ touches the reciprocal quadric $Sq(f_1+xf_2)q=0$. Thus we find

$$Sa(f_1+xf_2)aSb(f_1+xf_2)b-(Sa(f_1+xf_2)b)^2=0,$$

or simply $$(x-u)(x-v)=0,$$

so that the line touches the quadrics touched by the planes.]

Ex. 22. Show that the equation of the tangent cone from the extremity of the vector ρ to the quadric

$$Sq(f_1+xf_2)^{-1}q=0$$

may be written in the form

$$S\tau(\theta_1+x\theta_2)^{-1}\tau=0, \text{ where } \theta_n\lambda=\phi_n\lambda+e_n\rho S\rho\lambda-\epsilon_n S\rho\lambda-\rho S\epsilon_n\lambda$$

in the notation of Ex. 1, n being equal to 1 or 2.

[The condition that the line of intersection of the planes $Saq=0$, $Sbq=0$ should touch the quadric $Sqf^{-1}q=0$ may by the last example be written in the form

$$(e+2S\epsilon\alpha+S\alpha\phi\alpha)(e+2S\epsilon\beta+S\beta\phi\beta)-(e+S\epsilon(\alpha+\beta)+S\alpha\phi\beta)^2=0,$$

where $a=1+\alpha$, $b=1+\beta$. This reduces to

$$-SV\alpha\beta\psi V\alpha\beta+2S\epsilon\phi(\alpha-\beta)V\alpha\beta+eS(\alpha-\beta)\phi(\alpha-\beta)-S\epsilon(\alpha-\beta)^2=0,$$

and if the line of intersection of the planes is parallel to τ and if ρ is the vector to a point on it, we may take $V\alpha\beta=\tau$, $\beta-\alpha=-V\rho\tau$ (see p. 40, Ex. 4), or $\beta-\alpha=-V\rho V\alpha\beta$. Substituting this last expression for $\beta-\alpha$, we find that the condition becomes

$$SV\alpha\beta\{V\phi\alpha\phi\beta-V\phi\alpha(\epsilon S\rho\beta+\rho S\epsilon\beta)-V(\epsilon S\rho\alpha+\rho S\epsilon\alpha)\phi\beta$$
$$+eV\phi\alpha\rho S\rho\beta+eV\rho\phi\beta S\rho\alpha-V\epsilon\rho(S\epsilon\alpha S\rho\beta-S\epsilon\beta S\rho\alpha)\}=0$$

or $$SV\alpha\beta V(\phi\alpha-\epsilon S\rho\alpha-\rho S\epsilon\alpha+e\rho S\rho\alpha)(\phi\beta-\epsilon S\rho\beta-\rho S\epsilon\beta+e\rho S\rho\beta)=0.$$

In this transformation we make use of the fact that

$$SV\epsilon V\alpha\beta\phi V\rho V\alpha\beta=SV\rho V\alpha\beta\phi V\epsilon V\alpha\beta$$

in order to have the function in the last expression self-conjugate. If then

$$\theta\lambda=\phi\lambda-\epsilon S\lambda\rho-\rho S\lambda\epsilon+eS\lambda\rho,$$

the condition becomes $S\tau\theta^{-1}\tau=0$, and putting $f=f_1+xf_2$, and therefore $\theta=\theta_1+x\theta_2$, the result required is obtained.]

Ex. 23. If p is any point; p_1, p_2, p_3 the reciprocals of the tangent planes to the three confocals (parameters t_1, t_2, t_3) which pass through the point; show that the tangent cone to any other confocal (parameter t) is

$$\frac{x_1{}^2S\,.\,p_1{}^2}{t-t_1}+\frac{x_2{}^2S\,.\,p_2{}^2}{t-t_2}+\frac{x_3{}^2S\,.\,p_3{}^2}{t-t_3}=0,$$

where any point q is expressed in the form $xp+x_1p_1+x_2p_2+x_3p_3$.

[The condition that the line $p+uq$ should touch the confocal t is

$$Sq(f+t)^{-1}qSp(f+t)^{-1}p-(Sq(f+t)^{-1}p)^2=0, \text{ or } Sqhq=0$$

if h is the linear function defined by

$$hq=(f+t)^{-1}q\,Sp(f+t)^{-1}p-(f+t)^{-1}pSq(f+t)^{-1}p.$$

Substituting in turn p, p_1 $(=(f+t_1)^{-1}p)$, p_2, and p_3 for q, we find $hp=0$,

$$hp_1=(t-t_1)^{-1}p_1Sp(f+t)^{-1}p, \text{ etc.,}$$

because we have

$$h(f+t_1)^{-1}p=(f+t)^{-1}(f+t_1)^{-1}\{p\mathrm{S}p(f+t)^{-1}p-(f+t_1)p\mathrm{S}p(f+t_1)^{-1}(f+t)^{-1}p\},$$

which reduces to

$$h(f+t_1)^{-1}p=(t-t_1)^{-1}(f+t_1)^{-1}p\,\mathrm{S}p(f+t)^{-1}p$$

since $\quad \mathrm{S}p(f+t)^{-1}p-\mathrm{S}p(f+t_1)^{-1}p=(t_1-t)\mathrm{S}p(f+t_1)^{-1}(f+t)^{-1}p.$

The equation $\mathrm{S}qhq=0$ reduces to the required form since $\mathrm{S}p_1p_2=0$, etc.]

Ex. 24. Find the equation of the tangent line developable of the quadrics $\mathrm{S}\,.\,q^2=0$, $\mathrm{S}qfq=0$.

[If p is the point of contact of a tangent line pq to the common curve, the four conditions $\mathrm{S}\,.\,p^2=0$, $\mathrm{S}pq=0$, $\mathrm{S}pfp=0$, $\mathrm{S}pfq=0$, show that

$$(p,\, q,\, fp,\, fq)=0, \quad \text{or that} \quad (f+x)p=(f+y)q,$$

where x and y are two scalars. Substituting for p in the conditions of contact, we find four relations in q, x and y, which are easily seen to be equivalent to three. The second condition gives $A=\mathrm{S}q(f+x)^{-1}(f+y)q=0$; and because the first and third combine into $\mathrm{S}p(f+y)p=0$, they give

$$\mathrm{S}q(f+x)^{-2}(f+y)q=0, \quad \text{or} \quad \frac{\partial}{\partial x}\mathrm{S}q(f+x)^{-1}(f+y)q=0.$$

Again the second and fourth give

$$\mathrm{S}p(f+x)q=0, \quad \text{or} \quad B=\mathrm{S}q(f+y)q=0.$$

To eliminate x and y we have therefore to equate to zero the discriminant of A with respect to x and to employ the condition $B=0$. On expansion A becomes

$$\mathrm{S}q(F+xG+x^2H+x^3)(f+y)q=0,$$

and as $B=0$, this reduces to the quadratic

$$\mathrm{S}q\,(F+xG+x^2H)(f+y)q=0,$$

and the discriminant equated to zero gives

$$4\mathrm{S}qH(f+y)q\,\mathrm{S}qF(f+y)q-(\mathrm{S}qG(f+y)q)^2=0.$$

Putting for y its value in terms of q the required equation is obtained.]

Ex. 25. A plane is drawn through the line ab, and through the line cd the plane is drawn which is conjugate to this with respect to the quadric $\mathrm{S}qfq=0$. The locus of the intersection of the plane is

$$\mathrm{S}[qab]f^{-1}[qcd]=0.$$

[If q is a point on the intersection, $[qab]$ and $[qcd]$ are the symbols of the two planes. The equation may be transformed by Ex. 5, Art. 146.]

ART. **153**. A linear quaternion function has in general sixteen square roots quite analogous to the square roots of a linear vector function. A function and its square roots have the same united points, and the latent roots of the derived functions are the square roots of those of the original, there being sixteen different sets according to the choice of signs. (Compare p. 99.)

In analogy with the reduction of a linear vector function to the product of a conical rotator and of a self-conjugate function, we may write

$$fp=f_sf_tp, \quad f'p=f_t'f_sp, \quad \text{where } f_s=f_s' \text{ and } f_tf_t'=1, \quad \ldots\text{(I.)}$$

since if we take f_s to be a square root of the product ff' we must have $f_t f_t' = 1$ because

$$f_s^2 = ff' = f_s f_t f_t' f_s;$$

and thus we have

$$f_s = (ff')^{\frac{1}{2}}, \quad f_t = (ff')^{-\frac{1}{2}} f. \quad \text{.....................(II.)}$$

It appears on counting the constants that f_t is not a conical rotator, there being sixteen constants in f and only ten in the self-conjugate function f_s, so that there must be six in f_t. Considered geometrically the function f_t converts the unit sphere into itself and leaves unchanged conditions of conjugation with respect to that sphere, because

$$Sf_t a f_t b = 0 \quad \text{if} \quad Sab = 0.$$

Further, because $f_t = f_t'^{-1}$ transformation of symbols of planes effected by the function f_t is identical with that of points (Art. 149).

To study the nature of a function f_t which satisfies the relation

$$f_t f_t' = 1 = f_t' f_t \quad \text{or} \quad f_t = f_t'^{-1} \quad \text{or} \quad f_t' = f_t^{-1}, \quad \text{...........(III.)}$$

we shall endeavour to reduce the function to the form

$$f_t = f_u f_r, \quad \text{where} \quad f_u = f_u', \quad f_r = r(\quad)r^{-1}, \quad \text{...........(IV.)}$$

that is to the product of a self-conjugate function and a rotator. First we notice that if a function f_r, which satisfies the condition $f_r f_r' = 1$, converts a scalar into a scalar, it is a conical rotator, affected it may be with a minus sign. For if

$$f_r(1) = 1 = f_r'(1),$$

we have for all vectors ρ,

$$Sf_r\rho = S\rho f_r'(1) = S\rho = 0.$$

Thus $f_r\rho$ is a vector, and the mutual inclinations of vectors and their lengths remain unchanged after operation by f_r because

$$Sf_r\rho f_r'\rho = S\rho\rho'.$$

To effect the reduction (IV.), we notice that we must have

$$f_u^2 = 1, \quad f_t(1) = f_u(1), \quad \text{........................(V.)}$$

because $\quad f_t f_t' = f_u f_r f_r' f_u = f_u^2 \quad$ and $\quad f_t(1) = f_u f_r(1) = f_u(1).$

Let us now for the sake of symmetry introduce two quaternions a and b defined by the relations

$$1 + f_t(1) = a = 1 + f_u(1), \quad 1 - f_t(1) = b = 1 - f_u(1). \quad \text{......(VI.)}$$

These quaternions are known when the function f_t is given, and in virtue of (V.),

$$f_u a = a, \quad f_u b = -b, \quad Sab = 0, \quad \text{.................(VII.)}$$

so that a and b are united points of the function f_u.

Take any point c conjugate to the line ab, so that $Sac=0$, $Sbc=0$; and take the point d conjugate to the plane abc so that $Sda=0$, $Sdb=0$, $Sdc=0$. Then we may assume

$$f_u c=c,\ \ f_u d=-d, \qquad \text{(VIII.)}$$

and it is evident that all conditions (Art. 152, p. 276) are satisfied for the self-conjugation of the function f_u, and that $f_u^2 p=p$, where p is any point whatever. The function f_u is determined by the four conditions (VII.) and (VIII.), and the rotator f_r is given by $f_u^{-1}f_t$ or by its equivalent $f_u f_t$. It will be noticed that there is an infinite number of ways in which this reduction may be made, for the point c may be any point whatever on the reciprocal of the line ab. Also the function f_u has two line loci of united pointst—he line ac and the reciprocal line bd.

Thus we can in an infinite variety of ways reduce an arbitrary function f to the form

$$f=f_s f_u f_r,\ \text{ where }\ f_s=(ff')^{\frac{1}{2}},\ \ f_u^2=1,\ \ f_r=r(\quad)r^{-1}. \qquad \text{(IX.)}$$

As a simple example, consider the transformations which convert one quadric into another, or which change

$$Sqf_1q=0\ \text{ into }\ Spf_2p=0,\ \text{ where }\ p=fq. \qquad \text{(X.)}$$

We have

$$f_1=f'f_2f,\ \text{ whence }\ 1=f_t'f_t\ \text{ if }\ f=f_2^{-\frac{1}{2}}f_tf_1^{\frac{1}{2}}, \qquad \text{(XI.)}$$

and the function f_t is quite arbitrary subject to the condition $f_t'f_t=1$.

As another example we propose to show that the intersection of two quadrics is expressible in the form

$$q=(f+t)^{\frac{1}{2}}a, \qquad \text{(XII.)}$$

where f is a linear function, t a parameter and a a constant quaternion.

If this curve lies on the quadric $Sqf_1q=0$, the relation

$$S(f+t)^{\frac{1}{2}}af_1(f+t)^{\frac{1}{2}}a=Sa(f'+t)^{\frac{1}{2}}f_1(f+t)^{\frac{1}{2}}a=0$$

must be identically satisfied for all values of t. Now

$$f_1^{\frac{1}{2}}(f+t)^{\frac{1}{2}}f_1^{-\frac{1}{2}}=(f_1^{\frac{1}{2}}ff_1^{-\frac{1}{2}}+t)^{\frac{1}{2}},\ \ f_1^{-\frac{1}{2}}(f'+t)^{\frac{1}{2}}f_1^{\frac{1}{2}}=(f_1^{-\frac{1}{2}}f'f_1^{\frac{1}{2}}+t)^{\frac{1}{2}}, \qquad \text{(XIII.)}$$

as appears by squaring both members of each equation, so that the condition may be written

$$Saf_1^{\frac{1}{2}}(f_1^{-\frac{1}{2}}f'f_1^{\frac{1}{2}}+t)^{\frac{1}{2}}(f_1^{\frac{1}{2}}ff_1^{-\frac{1}{2}}+t)^{\frac{1}{2}}f_1^{\frac{1}{2}}a=0.$$

This becomes rational in t if the square roots involving t are identical, that is if

$$f_1^{-\frac{1}{2}}f'f_1^{\frac{1}{2}}=f_1^{\frac{1}{2}}ff_1^{-\frac{1}{2}}\ \text{ or }\ f'f_1=f_1f\ \text{ or if }\ f=f_1^{-1}f_2, \qquad \text{(XIV.)}$$

where f_2 is a self-conjugate function, the condition now becoming

$$Sa(f_2+tf_1)a=0,\ \text{ or }\ Saf_2a=0\ \text{ and }\ Saf_1a=0.$$

Finally,

$q=(f_1{}^{-1}f_2+t)^{\frac{1}{2}}a$, where $Saf_1a=0$, $Saf_2a=0$, $Saf_2f_1{}^{-1}f_2a=0$ (xv.) is the curve of intersection of the two quadrics $Sqf_1q=0$, $Sqf_2q=0$, because if we put $f_1{}^{-1}f_2=f_2{}^{-1}.f_2f_1{}^{-1}f_2$, and notice that $f_2f_1{}^{-1}f_2$ is a self-conjugate function, the conditions that the curve should lie on the second quadric are seen to be the second and third of the conditions (xv.). Thus a is one of the points of intersection of three known quadrics.

Ex. 1. Investigate the transformation of one quadric into another by first transforming to the unit sphere, then transforming the sphere into itself, and finally transforming the sphere into the second quadric.

[If $Sqf_1q=0$, $Sqf_2q=0$ are the two quadrics, the steps are

$$q_1=f_1^{\frac{1}{2}}q,\ q_2=f_tq_1,\ q_3=f_2^{-\frac{1}{2}}q_2,$$

where $f_tf_t'=1$.]

Ex. 2. Under what conditions can a function f be formed so that for all points q and q' we shall have

$$Spf_1p'=Sqf_2q',\ \text{where}\ p=fq\ \text{and}\ p'=fq'\,?$$

(a) Find the function f when the conditions are satisfied.

[We must have $f'f_1f=f_2$ with the implied relation $f'f_1'f=f_2'$ connecting the conjugates of these functions. Hence

$$f_2^{-1}f_2'=f^{-1}f_1^{-1}f'^{-1}.f'f_1'f=f^{-1}.f_1^{-1}f_1'.f,$$

and therefore the latent roots of the function $f_2^{-1}f_2'$ must be identical with those of $f_1^{-1}f_1'$. For if a_2, b_2, c_2, d_2 are the united points of $f_2^{-1}f_2'$ and if t_1, t_2, t_3 and t_4 are the corresponding latent roots we have (see p. 100)

$$f_1^{-1}f_1'.fa_2=t_1fa_2,\ \text{etc.}$$

Further if x, y, z and w are certain scalars and if a_1, b_1, c_1, d_1 are the united points of $f_1^{-1}f_1'$, we must have $fa_2=xa_1$, $fb_2=yb_1$, $fc_2=zc_1$, $fd_2=wd_1$; and because $f'f_1f=f_2$ we have $n^2n_1=n_2$, where n, n_1 and n_2 are the fourth invariants of f, f_1 and f_2. But

$$n(a_2b_2c_2d_2)=xyzw(a_1b_1c_1d_1),\ \text{or}\ (a_2b_2c_2d_2)\sqrt{n_2}=xyzw(a_1b_1c_1d_1)\sqrt{n_1},$$

and subject to this condition x, y, z and w are arbitrary, and the function f involves these arbitrary constants and is given by

$$fq.(a_2b_2c_2d_2)=-\Sigma xa_1(b_2c_2d_2q).]$$

Ex. 3. Under what conditions can two quadrics $Sqf_1q=0$, $Sqf_2q=0$ be transformed into two others $Sqf_3q=0$, $Sqf_4q=0$?

[This is nearly the same as the last example. We must have $f'f_1f=uf_3$, $f'f_2f=vf_4$, where u and v are scalars, and hence $f^{-1}f_2^{-1}f_1f=uv^{-1}f_4^{-1}f_3$, so that the latent roots of $f_2^{-1}f_1$ and of $f_4^{-1}f_3$ must be proportional. In the same way we obtain the conditions that a linear complex and a quadric should be simultaneously converted into a linear complex and a quadric.]

Ex. 4. A twisted cubic $q=(abcd\between t,1)^3$ may be converted into another $q'=(a'b'c'd'\between t',1)^3$ with arbitrary correspondence of points.

[Assuming $t'=\dfrac{u+tv}{u't+v'}$, where u, v, u' and v' are arbitrary scalars, we establish a homography connecting the points on one cubic with those on the other, and if we equate corresponding powers of t in the relation

$$f.(abcd\between t,1)^3=(a'b'c'd'\between ut+v,u't+v')$$

we have four relations which determine the function f.]

Ex. 5. Prove that

$$q=\sqrt{\{(f+x)(f+y)(f+z)\}}\,.\,e, \text{ where } Se^2=Sefe=Sef^2e=0,$$

represents a confocal of a generalized system when two of the parameters x, y, z vary; the intersection of two confocals when only one parameter varies; and a point common to the three confocals corresponding to given values of the parameters. (See p. 124.)

Ex. 6. The generalized confocals are inscribed to the developable of which

$$q=(f+x)^{\frac{3}{2}}e$$

is the cuspidal edge.

[The line of the developable corresponding to x is $q=(f+u)(f+x)^{\frac{1}{2}}e$; the osculating plane is $q=(f+u)(f+v)(f+x)^{-\frac{1}{2}}e$; the symbol of this plane is $[(f+x)^{-\frac{1}{2}}e,\ f(f+x)^{-\frac{1}{2}}e,\ f^2(f+x)^{-\frac{1}{2}}e]$, or $(f+x)^{\frac{1}{2}}[e,\ fe,\ f^2e]$, or simply $p=(f+x)^{\frac{1}{2}}e$. This plane touches every confocal.]

Ex. 7. Eight generators of the circumscribing developable are generators of an arbitrary quadric of the confocal system.

[The line $(f+u)(f+x)^{\frac{1}{2}}e$ is a generator of $Sq(f+x)^{-1}q=0$, and this is one of eight corresponding to the eight values of e deduced from the conditions of Ex. 5.]

Ex. 8. Eight rays of the complex of lines joining points to their correspondents in an arbitrary linear transformation are generators of an arbitrary quadric.

[The equation of a ray of the complex is $q=(f+u)a$, where a is arbitrary. This is a generator of the quadric $Sqf_1q=0$ if $Saf_1a=0$, $Sa(f'f_1+f_1f)a=0$, $Saf'f_1fa=0$. This is the generalization of Hamilton's theory of the umbilical generatrices.]

Ex. 9. The reciprocal of the developable generated by the tangents to the curve

$$q=(f+t)^m a \text{ is } p=(f'+t)^{2-m}b, \text{ where } b=[a, fa, f^2a]$$

and where m is a given scalar.

Ex. 10. The family of curves $q=(f+t)^m a$ includes the right line, the conic, the twisted cubic, the quartic intersection of two quadrics, the excubo quartic and the cuspidal edge of the developable circumscribed to two quadrics; the corresponding values of m are 1, 2, -1 or $+3$, $\frac{1}{2}$, 4 and $\frac{3}{2}$.

Ex. 11. The centres of generalized curvature at a point on the quadric $Sq(f+x)^{-1}q=0$ are

$$c=\frac{f+y}{f+x}q \text{ and } c'=\frac{f+z}{f+x}q,$$

where y and z are the parameters of the confocals which pass through the point q.

[The point $c=(f+u)(f+x)^{-1}q$ is situated on the generalized normal at q (Ex. 12, p. 279), and if this point remains stationary, that is if it is the point of intersection of consecutive normals,

$$dc=cdv=(f+u)(f+x)^{-1}q\,dv=(f+u)(f+x)^{-1}dq+(f+x)^{-1}q\,du,$$

since as c is stationary dc and c must represent the same point so that $dc=cdv$, where dv is some small scalar. This condition may be replaced by $dq=(f+u)^{-1}(f+w)q\,dv$, where w is a scalar, and operating by $S(f+x)^{-1}q$, we find almost exactly as in Art. 82, Ex. 4, p. 122, the required result.]

Ex. 12. The surface of centres of the quadric x is represented by

$$q=(f+x)^{-\frac{1}{2}}(f+y)^{\frac{3}{2}}(f+z)^{\frac{1}{2}}e\,; \quad Se^2=Sefe=Sef^2e=0.$$

Ex. 13. The differential equation of right lines on the surface

$$Sq(f+x)^{-1}q=0$$

is

$$\frac{dy}{\sqrt{n(y)}}\pm\frac{dz}{\sqrt{n(z)}}=0,$$

where $n(y)$ is the fourth invariant of $f+y$.

[The differential of $q=\sqrt{\{(f+x)(f+y)(f+z)\}}e$ is

$$dq=\tfrac{1}{2}\cdot\left(\frac{dy}{f+y}+\frac{dz}{f+z}\right)\cdot\sqrt{\{(f+x)(f+y)(f+z)\}}e\,;$$

and the differential equation of right lines on the surface is obtained by equating to zero

$$\begin{aligned}Sdq(f+x)^{-1}dq&=\tfrac{1}{4}Se\left(\frac{dy}{f+y}+\frac{dz}{f+z}\right)^2.(f+y)(f+z)e\\&=\tfrac{1}{4}\left(dy^2Se\frac{f+z}{f+y}e+dz^2Se\frac{f+y}{f+z}e\right)\\&=\tfrac{1}{4}(z-y).\{dy^2Se(f+y)^{-1}e-dz^2Se(f+z)^{-1}e\}.\end{aligned}$$

Now $Se(f+y)^{-1}e=n(y)^{-1}Se(F+yG+y^2H+y^3)e=n(y)^{-1}SeFe$ in virtue of the conditions satisfied by e.]

Ex. 14. The differential equation of generalized geodesics on the surface is

$$\sqrt{\frac{y-x}{n(y)(y-w)}}\,.\,dy\pm\sqrt{\frac{z-x}{n(z)(z-w)}}\,.\,dz=0,$$

where w is a constant of integration.

[A generalized geodesic is a curve whose osculating plane contains the pole of the tangent plane with respect to the quadric of reciprocation $(S.q^2=0)$. Thus $((f+x)^{-1}q,\ q,\ dq,\ d^2q)=0$ is the differential equation of a geodesic in terms of q and of its deriveds.

Writing this equation in the form $(f+x)^{-1}q+tq+udq+vd^2q=0$, where t, u and v are scalars, operating by Sq, Sdq, $S(f+x)^{-1}q$ and $S(f+x)^{-1}dq$, and observing that $Sdq(f+x)^{-1}dq+Sq(f+x)^{-1}d^2q=0$, we deduce

$$\frac{Sq(f+x)^{-2}dq}{Sq(f+x)^{-2}q}+\frac{Sdq(f+x)^{-1}d^2q}{Sdq(f+x)^{-1}dq}-\frac{S.q^2Sdqd^2q-SqdqSqd^2q}{S.q^2S.dq^2-Sqdq^2}=0.$$

This immediately integrates, and we find

$$Sq(f+x)^{-2}qSdq(f+x)^{-1}dq=s(S.q^2S.dq^2-Sqdq^2),$$

where s is a scalar constant. By the last example we have

$$Sdq(f+x)^{-1}dq=\tfrac{1}{4}(z-y)(n(y)^{-1}dy^2-n(z)^{-1}dz^2)SeFe,$$

and similarly

$$Sq(f+x)^{-2}q=(y-x)(z-x)n(x)^{-1}SeFe\,;\quad S.q^2=Sef^3e=-SeFe\,;\quad Sqdq=0,$$

$$S.dq^2=\tfrac{1}{4}(x-y)(z-y).n(y)^{-1}.dy^2.SeFe+\tfrac{1}{4}(x-z)(y-z).n(z)^{-1}.dz^2.SeFe.$$

Collecting these results and putting $x+sn(x)=w$, the required equation is obtained.]

ART. **154**. We shall now give a few examples relating to invariants of linear transformation and of quadric surfaces, and shall explain their geometrical import.*

By Art. 150 (v.), p. 273, the relation

$$((f-t)a,\ (f-t)b,\ (f-t)c,\ (f-t)d)$$
$$=(abcd)(n-n't+n''t^2-n'''t^3+t^4) \quad \text{(I.)}$$

is an identity for all scalars t and all quaternions a, b, c and d.

In this sense n, n', n'' and n''' are invariants, and every relation connecting them implies some peculiarity in the nature of the transformation effected by f. But there is a wider sense in which these four scalars are invariants. If n_1 and n_2 are the fourth invariants of two arbitrary functions f_1 and f_2, the relation

$$\left.\begin{array}{l}((f_1ff_2-tf_1f_2)a,\ (f_1ff_2-tf_1f_2)b,\ (f_1ff_2-tf_1f_2)c,\ (f_1ff_2-tf_1f_2)d)\\ \qquad =(abcd)n_1n_2(n-n't+n''t^2-n'''t^3+t^4),\end{array}\right\}\text{(II.)}$$

is evidently true since $(f_1p, f_1q, f_1r, f_1s)=n_1(pqrs)$, where p, q, r and s are any quaternions. Thus any relation implying a peculiarity of the function f and depending on its four *scalar* invariants, implies also a corresponding peculiarity in the mutual relations of the functions f_1ff_2 and f_1f_2, that is, in the relations of any pair of functions that can be reduced to the forms f_1ff_2 and f_1f_2. (See p. 98 and Ex. 3, p. 101.)

Ex. 1. If the function f transforms any tetrahedron $abcd$ into another $a'b'c'd'$ having its vertices on the faces of the original, the invariant n''' vanishes and an infinite number of tetrahedra possess the property. The converse is also true.

[The conditions are $(a'bcd)=0$, $(ab'cd)=0$, $(abc'd)=0$, $(abcd')=0$, and because $a'=fa$, etc., we find on addition that $n'''=0$. Let a, b and c be any arbitrary points, and let d be determined from the first three conditions. Then we have $n'''(abcd)=(abcd')$, so that if $n'''=0$, the point fd will lie on the face abc. More generally when $n'''=0$ there exists an infinite number of tetrahedra so that the tetrahedra derived from any one by the operation of the functions f_1ff_2 and f_1f_2 are related in the manner described.

If $n'=0$, the faces of the derived tetrahedra contain the vertices of the original.]

Ex. 2. The invariant n'''^2-2n'' vanishes whenever a tetrahedron $abcd$ is so related to its correspondent in the transformation, that the tetrahedron transformed from the correspondent has its vertices on the original.

[The sum of the squares of the latent roots of f is zero, or the first invariant of f^2 vanishes.]

Ex. 3. When the invariant

$$(n'''^2-4n'')^2-64n$$

vanishes it is possible to determine an infinite number of tetrahedra $(abcd)$

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* See *Phil. Trans.*, vol. 201, "Quaternions and Projective Geometry."

and their deriveds $(a'b'c'd')$ so that a tetrahedron can be inscribed to $abcd$ and circumscribed to $a'b'c'd'$.

[The sum of the square roots of the latent roots of f is zero, or the first invariant of one of its square roots $f^{\frac{1}{2}}$ vanishes.]

Ex. 4. If an *infinite* number of tetrahedra can be inscribed to one quadric surface and circumscribed to another, find the invariant relation.

[Let $abcd$ be the four vertices of a tetrahedron inscribed to the quadric $Sqf_1q=0$, and let the faces touch $Sqf_2q=0$ at the points $a'b'c'd'$. If $a'=fa$, etc., we have four equations of inscription $Saf_1a=0$, etc.; twelve equations of conjugation, $Sa'f_2b=0$, $Sb'f_2a=0$, etc., or $Saf'f_2b=0$, $Saf_2fb=0$, etc.; and four equations of contact $Sa'f_2a'=0$, or $Saf'f_2fa=0$. The equations of conjugation require f_2f to be self-conjugate, or $f_2f=f'f_2$; and the equations of contact may therefore be replaced by $Saf_2f^2a=0$, etc. Hence if the first invariant of f is zero and if $f_2f=f'f_2$, it is possible to inscribe in the quadric $Sqf_2f^2q=0$ and to circumscribe to $Sqf_2q=0$ an infinite number of tetrahedra. For when we assume two of the vertices a and b, we have to determine c and d to satisfy $(fa, b, c, d)=0$, $(a, fb, c, d)=0$, $(a, b, fc, d)=0$, $Scf_2f^2c=0$ and $Sdf_2f^2d=0$. The first three give d in terms of c, and on substitution in the fifth we have two equations in c, any solution of which will be applicable.

The quadrics $Sqf_2f^2q=0$ and $Sqf_2q=0$ possess therefore the required property, and so do the quadrics $Sqf_1q=0$ and $Sqf_2q=0$, if it is possible to find a function f for which $f_2f^2=f_1$, $f_2f=f'f_2$ and $n'''=0$. It is easy to see that the conditions are satisfied if the invariant of the last example vanishes for the function $f_2^{-1}f_1$.]

Ex. 5. If a tetrahedron circumscribed to $Sqf_2q=0$ is self-conjugate to $Sqf_3q=0$, the first invariant of the function $f_2^{-1}f_3$ vanishes.

[This is virtually proved in the last example, the function f_3 being f_2f.]

Ex. 6. When the invariant n'' vanishes, it is possible to determine an infinite number of tetrahedra $(abcd)$ and their deriveds $(a'b'c'd')$, so that each edge (ab) of one of the tetrahedra intersects the opposite edge $(c'd')$ of the correspondent.

[The invariant is $(abcd)n''=\Sigma(abc'd')$, and it manifestly vanishes if opposite edges intersect, that is if each of the six terms $(abc'd')$ vanishes. Conversely if $n''=0$, we may arbitrarily assume two of the points a and b. We have then to determine c and d to satisfy *five* conditions, $(abc'd')=0$, etc. Solving for d (Art. 146, Ex. 3, p. 269) from three of these and substituting in the remaining two, we get two equations quartic in c, and the point c lies on part of the curve of intersection of the quartic surfaces represented by these equations.]

Ex. 7. Find the locus of intersection of generators of a quadric which are the sides of a triangle self-conjugate to another quadric.

[If the quadrics are $Sqf_1q=0$, $Sqf_2q=0$, we may first reduce the second quadric to the sphere $Sq^2=0$ and the first to $Sqfq=0$ where $f=f_2^{-\frac{1}{2}}f_1f_2^{-\frac{1}{2}}$. If q is the intersection of the generators and a and b the remaining vertices of the triangle, the conditions are

$$Sqfq=Sqfa=Safa=Sqfb=Sbfb=0,\quad Saq=Sbq=Sab=0.$$

Now for the first invariant of f we have

$$n'''(a, b, q, fq)=(fa, b, q, fq)+(a, fb, q, fq)+(a, b, q, f^2q),$$

and the conditions require $(fa, b, q, fq)=0$ and $(a, fb, q, fq)=0$, because the four constituents of the first are reciprocal to a, while those of the second are reciprocal to b. Also $[a, b, q]=xfq$, and therefore the locus is

$$n'''Sqf^2q=Sqf^3q.]$$

Ex. 8. Three intersecting edges of a tetrahedron self-conjugate to one quadric touch another. Find the locus of the intersection.

[If the tetrahedron $qabc$ is self-conjugate to $Sq^2=0$, we have $wq=[abc]$, $xa=[bcq]$, $yb=[caq]$, $zc=[abq]$; and if the line qa touches $Sqfq=0$, the relation $SqfqSafa-Sqfa^2=0$ must be satisfied. This condition of contact may be written in the form $Sqfq(fa, b, c, q)-Sfqa(fq, b, c, q)=0$, and there are two similar conditions of contact obtained from this by cyclical interchange of a, b and c. Writing down the identical relation connecting a, b, c, q and fq, and utilizing the conditions, we find

$$Sfq\{fq(a, b, c, q)-q(a, b, c, fq)\}-Sqfq\{n'''(a, b, c, q)-(a, b, c, fq)\}=0\,;$$

and this reduces to $Sqf^2q-n'''Sqfq=0$, when the factor $S\,.\,q^2$ is discarded, remembering that $[abc]=wq$.]

Ex. 9. Each of three planes $Sqa=0$, $Sqb=0$, $Sqc=0$, mutually conjugate to $Sq^2=0$, touches one of the family of confocals $Sq(f+u)^{-1}q=0$. Find the locus of the intersection of the planes.

[The points q, a, b, c satisfy the conditions of the last example which do not depend on the function f. The conditions of contact are of the form

$$uSa^2+Safa=0 \quad \text{or} \quad u(abcq)+(fabcq)=0\,;$$

and hence

$$(u+v+w+n''')Sq^2-Sqfq=0$$

is the locus required.]

Ex. 10. The edge ab of a tetrahedron self-conjugate to $Sq^2=0$ touches the quadric $Sqfq=0$. The condition of contact may be reduced to

$$(fafbcd)=0,$$

and the invariant n'' vanishes if all the edges touch the quadric.

[By Ex. 6, Art. 146 and (VI.), and Ex. 1, Art. 145, this follows without trouble.]

Ex. 11. If the functions f_1, f_2, f_3, etc., are transformed by multiplying them by an arbitrary function f_x and into an arbitrary function f_y, the functions $f_1f_2^{-1}f_3$, $f_1f_2^{-1}f_3f_4^{-1}f_5$, etc., undergo the same transformation and may be said to be *covariant* with the original functions for this type of transformation.

(a) The function f_{123}, defined as the coefficient of $t_1t_2t_3$ in the identity

$$\Sigma t_1t_2t_3f_{123}[abc]=[\Sigma t_1f_1'^{-1}a,\ \Sigma t_1f_1'^{-1}b,\ \Sigma t_1f_1'^{-1}c],$$

where t_1, t_2, t_3, etc., are arbitrary scalars, is (to a scalar factor) covariant with the original functions.

(b) Examine the nature of the transformations the inverse and the conjugate functions undergo simultaneously with the original functions, and find the condition that self-conjugate properties may be preserved.

ART. **155**. Several important geometrical and numerical relations may be deduced from the identity

$$p_1(p_2p_3p_4p_5)+p_2(p_3p_4p_5p_1)+p_3(p_4p_5p_1p_2)$$
$$+p_4(p_5p_1p_2p_3)+p_5(p_1p_2p_3p_4)=0, \;\ldots\ldots(\text{I.})$$

in which p_n is a rational and integral homogeneous quaternion function of q of order m_n.

The scalar equations

$$(p_5 p_1 p_2 p_3)=0, \quad (p_1 p_2 p_3 p_4)=0 \quad \text{..............(II.)}$$

represent two surfaces of orders $\Sigma_1^5 m_1 - m_4$ and $\Sigma_1^5 m_1 - m_5$ respectively, and any point on their intersection satisfies the quaternion equation

$$[p_1 p_2 p_3]=0, \quad \text{..........................(III.)}$$

or else the three scalar equations

$$(p_2 p_3 p_4 p_5)=0, \quad (p_3 p_4 p_5 p_1)=0, \quad (p_4 p_5 p_1 p_2)=0. \quad \text{...(IV.)}$$

Hence we see that the curve of intersection of the surfaces (II.) breaks up into two parts, one of which is represented by (III.), while the other—the complementary curve—is common to the five surfaces (II.) and (IV.).

Now the order of the curve (III.) must be a symmetric function of m_1, m_2 and m_3, and that of the complementary curve must be a symmetric function of the five orders m_n. The sum of the orders is equal to the product of the orders of the surfaces (II.), that is, to

$$(\Sigma_1^5 m_1 - m_4)(\Sigma_1^5 m_1 - m_5) = \Sigma_1^5 m_1 m_2 + \Sigma_1^3 m_1^2 + \Sigma_1^3 m_1 m_2;$$

and accordingly the order of the curve (III.) and that of the complementary curve are respectively

$$m_{123} = \Sigma_1^3 m_1^2 + \Sigma_1^3 m_1 m_2 \quad \text{and} \quad m_c = \Sigma_1^5 m_1 m_2. \quad \text{.........(V.)}$$

Again the points common to the three surfaces (IV.) must either lie on the surfaces (II.) or else must satisfy the equation

$$(p_4 p_5)=0, \quad \text{............................(VI.)}$$

which requires $p_4 = u p_5$, where u is a scalar. In the former case the points lie on the complementary curve. When three surfaces have no common curve the number of their points of intersection is the product of their order; when they have a common curve, that curve counts for a definite number of points of intersection, and there are in general other points of intersection not on the curve.* Now the surfaces (IV.), if they had no common curve, would intersect in

$$\begin{aligned}(\Sigma_1^5 m_1 - m_1)&(\Sigma_1^5 m_1 - m_2)(\Sigma_1^5 m_1 - m_3)\\ &= \Sigma_1^5 m_1 \Sigma_1^5 m_1 m_2 - \Sigma_1^5 m_1 m_2 m_3 + (m_4 + m_5)\Sigma_1^5 m_1 m_2\\ &\qquad + m_4^3 + m_4^2 m_5 + m_4 m_5^2 + m_5^3\end{aligned}$$

common points, the number being transformed so as to exhibit it as a function of symmetric functions of the five orders and of symmetric functions of m_4 and m_5. The number of points satisfying (VI.) must be a symmetric function of m_4 and m_5

* Salmon, *Three Dimensions*, Art. 355.

alone. The number of points of intersection of the surfaces (IV.) absorbed by the complementary curve is (*Three Dimensions*, Art. 355) a linear function of the order and rank of the curve—and the order and rank must both be symmetric functions of the five orders. Hence the number of solutions of (VI.) is

$$t_{45}=m_4^3+m_4^2m_5+m_4m_5^2+m_5^3. \qquad \text{(VII.)}$$

In the next place, in order to find the rank and the number of apparent double points of the curve (III.), we notice that it meets the surface $(p_4p_5p_1p_2)=0$ in $m_{123}(\Sigma_1^5m_1-m_3)$ points. These points, as appears from (I.), are either solutions of $(p_1p_2)=0$ or points on the complementary curve. The number of intersections of (III.) with the complementary curve is therefore by (VII.)

$$\begin{aligned} t_{123}&=m_{123}(\Sigma_1^5m_1-m_3)-t_{12} \\ &=m_{123}\Sigma_1^5m_1-\Sigma_1^3m_1^3-\Sigma_1^3m_1^2m_2-m_1m_2m_3. \qquad \text{(VIII.)} \end{aligned}$$

Employing the relation $r+t=m(\mu+\nu-2)$ of Salmon's *Three Dimensions*, Art. 346, connecting the rank r and the number of intersections t of a curve of order m and its complementary on two surfaces of orders μ and ν, we find for the surfaces (II.) of orders $\Sigma_1^5m_1-m_4$ and $\Sigma_1^5m_1-m_5$ that the rank of the curve (III.) is

$$r_{123}=-t_{123}+m_{123}(2\Sigma_1^5m_1-m_4-m_5-2),$$

which reduces by (VIII.) to

$$\begin{aligned} r_{123}&=m_{123}(\Sigma_1^3m_1-2)+\Sigma_1^3m_1^3+\Sigma_1^3m_1^2m_2+m_1m_2m_3 \\ &=m_1m_2m_3-3\Sigma_1^3m_1\Sigma_1^3m_1m_2+2(\Sigma_1^3m_1)^3 \\ &\quad -2((\Sigma_1^3m_1)^2-\Sigma_1^3m_1m_2). \qquad \text{(IX.)} \end{aligned}$$

In the next place, to find the number (h_{123}) of apparent double points of the curve (III.), we have (*Three Dimensions*, Art. 346),

$$h_{123}=\tfrac{1}{2}m_{123}(m_{123}-1)-\tfrac{1}{2}r_{123}. \qquad \text{(X.)}$$

The rank (r_c) of the complementary curve is determined by

$$r_c=-t_{123}+m_c(2\Sigma_1^5m_1-m_4-m_5-2),$$

and this may be reduced to

$$r_c=\Sigma_1^5m_1\Sigma_1^5m_1m_2+\Sigma_1^5m_1m_2m_3-2\Sigma_1^5m_1m_2, \qquad \text{(XI.)}$$

and the number of apparent double points is

$$h_c=\tfrac{1}{2}m_c(m_c-1)-\tfrac{1}{2}r_c.$$

We may denote the complementary curve by the symbol

$$((p_1p_2p_3p_4p_5))=0, \qquad \text{(XII.)}$$

which is intended to denote that the points of the curve satisfy every equation obtained by omitting one symbol. Similarly,

$$(((p_1p_2p_3p_4p_5p_6)))=0 \qquad \text{(XIII.)}$$

may be taken to denote the points which satisfy the surfaces obtained by omitting two symbols. These points lie on the curve (XII.) and also on the surface $(p_1 p_2 p_3 p_6)=0$. But the intersection of the curve and the surface includes the points t_{123} on the curve $[p_1 p_2 p_3]=0$. Omitting these, the number of points is

$$m_c(m_1+m_2+m_3+m_6)-t_{123}=\Sigma_1{}^6 m_1 m_2 m_3. \quad \text{........(XIV.)}$$

Ex. 1. The curve $[q, fq, a]=0$, where f is a linear function, is a cubic; its rank is 4 and the number of its apparent double points is 1.

Ex. 2. The curve $[f_1 q, f_2 q, f_3 q]=0$ is a sextic of rank 16 and with 7 apparent double points. It is the locus of points that can be destroyed by functions of the system $t_1 f_1+t_2 f_2+t_3 f_3$, and the locus of united points of functions of the system

$$\frac{t_1 f_1+t_2 f_2+t_3 f_3}{u_1 f_1+u_2 f_2+u_3 f_3},$$

where t and u are scalars.

Ex. 3. The surface $(f_1 q, f_2 q, f_3 q, f_4 q)=0$

is the locus of united points of a family of linear functions.

(a) When the functions are self-conjugate, it is the Jacobian of four quadrics.

Ex. 4. The curve $((f_1 q, f_2 q, f_3 q, f_4 q, f_5 q))=0$

is of the tenth order and its rank is 40.

(a) The Jacobians of sets of four out of five given quadrics have a common curve, and the Jacobians of sets of four out of six quadrics have twenty common points.

ART. **156.** If Q is any homogeneous and *scalar* function of q of order m, but not necessarily rational or integral, the equation

$$Q=0 \quad \text{..(I.)}$$

represents a surface.

We shall write the differential of the function Q in the form

$$dQ=m\mathrm{S}p\,dq, \quad \text{............................(II.)}$$

where p is a homogeneous function of q of order $m-1$. By Euler's theorem concerning homogeneous functions, we see by (II.) that

$$Q=\mathrm{S}pq=P, \quad \text{............................(III.)}$$

where P is the function of p into which Q transforms when q expressed as a fraction of p is substituted in Q, for we may regard q as a function of p since p is a determinate function of q.

Again we shall write generally for the differential of p,

$$dp=(m-1)f_q dq, \quad \text{.....................(IV.)}$$

where $f_q dq$ is a linear function of dq and where the constituents of f_q involve q in the order $m-2$; and by Euler's theorem we have

$$p=f_q q. \quad \text{..............................(V.)}$$

This function f_q is self-conjugate, as we have shown in a more general case (Art. 60 (IV.), p. 79).

Now if we differentiate (III.) we have

$$dQ = Spdq + Sqdp = dP, \qquad \text{(VI.)}$$

and on comparison with (II.) we see that

$$dP = nSqdp, \text{ where } (n-1)(m-1)=1, \qquad \text{(VII.)}$$

and it is easy to verify that n is the order in which p is involved in p.

We shall also write generally for the differential of q expressed as a function of p,

$$dq = (n-1)f_p dp, \qquad \text{(VIII.)}$$

and the function f_p is also self-conjugate and involves p in the order $n-2$ in its constitution. Thus for any differential by (IV.) and (VIII.) we have

$$dp = (m-1)f_q dq = (m-1)(n-1)f_q f_p \,.\, dp = f_q f_p \,.\, dp \quad \text{(IX.)}$$

by (VII.), and accordingly

$$f_q f_p = 1 = f_p f_q, \qquad \text{(X.)}$$

or one function produces on an arbitrary quaternion the same effect as the reciprocal of the other. In particular, applying Euler's theorem to (VIII.) as we have already applied it to (IV.), we obtain the relations

$$p = f_q q = f_p^{-1} q, \quad q = f_p p = f_q^{-1} p. \qquad \text{(XI.)}$$

When dq instead of being perfectly arbitrary satisfies

$$dQ = 0, \text{ or } Spdq = 0 \text{ where } Q = 0, \qquad \text{(XII.)}$$

dq represents some point in the tangent plane at q, and p is the symbol of the tangent plane or the reciprocal of the plane with respect to the auxiliary quadric. The equation $P=0$ is that of the reciprocal of the surface. The relations of reciprocity are clearly exhibited by the equations (compare (II.), (III.) and (VI.))

$$Spdq = 0, \; Sqdp = 0, \; dP = 0, \; P = 0 \text{ if } dQ = 0, \; Q = 0; \quad \text{(XIII.)}$$

$$-Sdpdq = Spd^2q = Sqd^2p, \; d^2P = 0 \text{ if also } d^2Q = 0. \quad \text{(XIV.)}$$

Consecutive tangent planes at q and $q+dq$ intersect in the line common to the planes

$$Spr = 0, \; Sdpr = 0, \qquad \text{(XV.)}$$

r being the current point, and if $q+d'q$ is a consecutive point on this edge we have the group of relations

$$Spq = 0, \; Spdq = 0, \; Spd'q = 0, \; Sdpd'q = 0,$$
$$Sqdp = 0, \; Sqd'p = 0, \; Sd'pdq = 0, \qquad \text{(XVI.)}$$

remembering that in general $Sdpd'q = Sd'pdq$ because f_q is self-conjugate. Hence to conjugate* tangents (qdq and $qd'q$) on the surface correspond conjugate tangents on the reciprocal, and the reciprocal of a tangent to the surface is the correspondent of the conjugate tangent, for we have $S(p+xd'p)(q+ydq)=0$.

The differential equation of the asymptotic lines is

$$Sdpdq=0, \qquad \text{(XVII.)}$$

these lines being their own conjugates.

The differential equation of lines of curvature is

$$(pqdpdq)=0, \qquad \text{(XVIII.)}$$

for this is the condition that consecutive generalized normals should intersect. If c is a centre of curvature, we have

$$c=q+tp,\ dc=(1+tf_q)dq+pdt=(q+tp)du, \qquad \text{(XIX.)}$$

where du and dt are some small scalars. (Compare Art. 153, Ex. 11.) Hence as $p=f_qq$ we obtain the relation

$$qdu-dq=(f_q^{-1}+t)^{-1}qdt;$$

and operating by Sf_qq we get

$$Sqf_q(f_q^{-1}+t)^{-1}q=0 \text{ or } Sq(f_q^{-1}+t)^{-1}q=0, \qquad \text{(XX.)}$$

since $f_q(f_q^{-1}+t)^{-1}=t^{-1}\{f_q-(f_q^{-1}+t)^{-1}\}$ and $Sqf_qq=0$.

The theory of generalized curvature is thus connected with that of the generalized confocals. The scalar t is the parameter of one of the confocals $Sr(f_p+t)^{-1}r=0$ which pass through q, r being the current variable. The confocal $t=0$ is $Srf_qr=0$.

The roots of this equation in t determine the centres of curvature, and because in terms of $f_p(=f_q^{-1})$ it becomes

$$Sq(F_p+tG_p+t^2H_p+t^3)q=0 \text{ or } Sq(G_p+t^2H_p+t^3)q=0 \qquad \text{(XXI.)}$$

(since $F_p=n_pf_p^{-1}=n_pf_q$ and $Sqf_qq=0$) after discarding the factor t, it reduces to a quadratic and gives two values of t.

Ex. 1. The points having common polar planes with respect to two surfaces satisfy the equation

$$(p_1p_2)=0;$$

the points having collinear polar planes with respect to three surfaces lie on the curve

$$[p_1p_2p_3]=0;$$

the points having concurrent polar planes with respect to four surfaces generate the Jacobian

$$(p_1p_2p_3p_4)=0;$$

the points having concurrent polar planes with respect to five surfaces lie on the curve

$$((p_1p_2p_3p_4p_5))=0;$$

*Consecutive tangent planes intersect in the tangent line *conjugate* to that joining their points of contact.

and the points having concurrent polar planes with respect to six surfaces satisfy the equation

$$(((p_1 p_2 p_3 p_4 p_5 p_6)))=0;$$

provided we write generally $\mathrm{d}Q_n = m_n \mathrm{S} p_n \mathrm{d}q$, where $Q_n=0$ is the equation of one of the surfaces.

Ex. 2. To find the osculating plane at a point on the curve of intersection of two given surfaces.

[The osculating plane must pass through the intersection of the tangent planes at the point q, and its equation must be of the form

$$\mathrm{S}p_1 r + t\mathrm{S}p_2 r = 0,$$

where $\mathrm{S}p_1 r=0$ and $\mathrm{S}p_2 r=0$ are the tangent planes. We have identically

$$\mathrm{S}p_1 q = \mathrm{S}p_2 q = \mathrm{S}p_1 \mathrm{d}q = \mathrm{S}p_2 \mathrm{d}q = 0,$$

and by (XIV.) the scalar t is determined by the condition

$$\mathrm{Sd}p_1 \mathrm{d}q + t\mathrm{Sd}p_2 \mathrm{d}q = 0,$$

so that the osculating plane is

$$\mathrm{S}p_1 r \mathrm{Sd}p_2 \mathrm{d}q - \mathrm{S}p_2 r \mathrm{Sd}p_1 \mathrm{d}q = 0.$$

This has now to be simplified. Assuming a quaternion a satisfying $\mathrm{S}a\mathrm{d}q=0$, we have $\mathrm{d}q=[p_1 p_2 a]$. Also $\mathrm{d}p_1=(m_1-1)f_1\mathrm{d}q$, $\mathrm{d}p_2=(m_2-1)f_2\mathrm{d}q$, and accordingly

$$\mathrm{Sd}p_1\mathrm{d}q=(m_1-1)\,\mathrm{S}[p_1p_2a]f_1[p_1p_2a]=(m_1-1)\,\mathrm{S}[p_1p_2a][qF_1p_2f_1^{-1}a],$$

since $f'[abc]=[f^{-1}aFbf^{-1}c]$. By Art. 146, Ex. 5, this becomes

$$-\mathrm{Sd}p_1\mathrm{d}q=(m_1-1)\begin{vmatrix} \mathrm{S}p_1q & \mathrm{S}p_1F_1p_2 & \mathrm{S}p_1f_1^{-1}a \\ \mathrm{S}p_2q & \mathrm{S}p_2F_1p_2 & \mathrm{S}p_2f_1^{-1}a \\ \mathrm{S}aq & \mathrm{S}aF_1p_2 & \mathrm{S}af_1^{-1}a \end{vmatrix}=(m_1-1)\begin{vmatrix} 0 & 0 & \mathrm{S}aq \\ 0 & \mathrm{S}p_2F_1p_2 & \mathrm{S}p_2f_1^{-1}a \\ \mathrm{S}aq & \mathrm{S}aF_1p_2 & \mathrm{S}af_1^{-1}a \end{vmatrix}$$

$$=-(m_1-1)\mathrm{S}aq^2\mathrm{S}p_2F_1p_2.$$

Hence the osculating plane is

$$(m_2-1)\mathrm{S}p_1r\mathrm{S}p_1F_2p_1-(m_1-1)\mathrm{S}p_2r\mathrm{S}p_2F_1p_2=0.]$$

ART. **157**. If we use the notation d_a to denote that the differential of q is equal to a quaternion a, we shall have for the k^{th} polar of a with respect to the surface $Q=0$,

$$\mathrm{d}_a{}^k Q = 0 \quad \text{where} \quad \mathrm{d}a=0, \qquad \text{(I.)}$$

and if m is the order of the surface, we may write the equation of the k^{th} polar in the form

$$\mathrm{d}_a{}^k \mathrm{d}_r{}^{m-k} Q = 0, \qquad \text{(II.)}$$

the quaternion r being now the variable point, and r being regarded as constant in performing the differentiations indicated.

If we write

$$\mathrm{d}_a Q = \mathrm{S}ap, \qquad \text{(III.)}$$

we may consider the quaternion p to be derived from the scalar Q by an operator D analogous to Hamilton's operator ∇, and we shall have generally and symbolically,

$$\mathrm{D}=\frac{[bcd]\mathrm{d}_a-[acd]\mathrm{d}_b+[abd]\mathrm{d}_c-[abc]\mathrm{d}_d}{(abcd)}; \qquad \text{(IV.)}$$

and in particular when $q=w+ix+jy+kz$, $a=1$, $b=i$, $c=j$ and $d=k$, we have

$$\mathrm{D}=\frac{\partial}{\partial w}-i\frac{\partial}{\partial x}-j\frac{\partial}{\partial y}-k\frac{\partial}{\partial z}=\frac{\partial}{\partial w}-\nabla. \quad\text{......}(\text{V.})$$

In this notation (I.) and (II.) become

$$(\mathrm{S}a\mathrm{D})^k\,.\,Q=0,\quad (\mathrm{S}a\mathrm{D})^k(\mathrm{S}r\mathrm{D})^{m-k}\,.\,Q=0. \quad\text{......}(\text{VI.})$$

We may also formally identify our notation with Aronhold's symbolic notation by writing the second of these expressions in the forms

$$(\mathrm{S}ae)^k(\mathrm{S}re)^{m-k}=0 \quad\text{or}\quad e_a{}^k e_r{}^{m-k}=0, \quad\text{......}(\text{VII.})$$

where e is a symbolic quaternion devoid of meaning unless it enters into a term homogeneous in e and of order m, and where $e_r=\mathrm{S}er$.

There is thus a considerable latitude in the choice of an appropriate notation for the investigation of projective properties of curves and surfaces.

Ex. 1. In investigations which involve differentials of the third order of the equation of an arbitrary surface of order m, we may write

$$\mathrm{d}_aQ=m\mathrm{S}pa,\quad \mathrm{d}_b\mathrm{d}_aQ=m(m-1)\mathrm{S}bfa,\quad \mathrm{d}_c\mathrm{d}_b\mathrm{d}_aQ=m(m-1)(m-2)\mathrm{S}cf_2(a,b)$$

with liberty to transpose in any way the quaternions a, b, c, the function $f_2(a,b)$ being a *bilinear* function of a and b (compare Art. 60).

(a) In terms of the operator D,

$$p=\frac{1}{m}\mathrm{D}\,.\,Q,\quad fa=\frac{1}{m(m-1)}\,.\,\mathrm{DS}a\mathrm{D}\,.\,Q,\quad f_2(a,b)=\frac{1}{m(m-1)(m-2)}\mathrm{DS}a\mathrm{D}\,\mathrm{S}b\mathrm{D}\,.\,Q.$$

(b) We may also write

$$Q=\mathrm{S}eq^3,\quad \mathrm{d}_aQ=m\mathrm{S}ea\mathrm{S}eq^2,\quad \mathrm{d}_b\mathrm{d}_aQ=m(m-1)\mathrm{S}eb\mathrm{S}ea\mathrm{S}eq,$$
$$\mathrm{d}_c\mathrm{d}_b\mathrm{d}_aQ=m(m-1)(m-2)\mathrm{S}ec\mathrm{S}eb\mathrm{S}ea,$$

where e is a symbolic quaternion devoid of meaning unless it occurs thrice in a term.

(c) We have

$$p=fq=f_2(q,q)=e\mathrm{S}eq^2\,;\quad fa=f_2(a,q)=e\mathrm{S}ea\mathrm{S}eq\,;\quad f_2(a,b)=e\mathrm{S}ea\mathrm{S}eb.$$

And when we differentiate fa totally we find

$$\mathrm{d}\,.\,fa=f\,.\,\mathrm{d}a+(m-2)f_2(a,\mathrm{d}q).$$

(d) The equation of the Hessian is

$$n=0 \quad\text{or}\quad (fa,fb,fc,fd)=0,$$

where n is the fourth invariant of f and where a, b, c and d are arbitrary points. It may also be expressed in the forms

$$(ee'e''e''')\mathrm{S}ea\,\mathrm{S}e'b\,\mathrm{S}e''c\,\mathrm{S}e'''d\,\mathrm{S}eq\,\mathrm{S}e'q\,\mathrm{S}e''q\,\mathrm{S}e'''q=0\,;$$
$$(ee'e''e''')^2\mathrm{S}eq\,\mathrm{S}e'q\,\mathrm{S}e''q\,\mathrm{S}e'''q=0\,;$$
$$\mathrm{S}a\mathrm{DS}b\mathrm{D}'\mathrm{S}c\mathrm{D}''\mathrm{S}d\mathrm{D}'''(\mathrm{D}Q,\mathrm{D}'Q',\mathrm{D}''Q'',\mathrm{D}'''Q''')=0\,;$$
$$(\mathrm{D}\mathrm{D}'\mathrm{D}''\mathrm{D}''')^2\,.\,QQ'Q''Q'''=0,$$

where e, e', e'', e''', etc., are equivalent symbols (compare Art. 147, p. 270).

Ex. 2. If $J=(p_1p_2p_3p_4)$ and $\mathrm{d}Q_n=m_n\mathrm{S}p_n\mathrm{d}q$, where Q_n is homogeneous and of order m_n in the variable q,

$$q\,.\,J=[p_2p_3p_4]\,.\,Q_1-[p_1p_3p_4]\,.\,Q_2+[p_1p_2p_4]\,.\,Q_3-[p_1p_2p_3]\,.\,Q_4.$$

(*a*) For an arbitrary differential, and for an arbitrary scalar m,

$$q\,.\,\mathrm{d}J=(m-1)J\,.\,\mathrm{d}q+\Sigma\pm Q_1\,.\,\mathrm{d}[p_2p_3p_4]+\Sigma\pm(m_1-m)[p_2p_3p_4]\mathrm{S}p_1\mathrm{d}q.$$

(*b*) If four surfaces have a common point, their Jacobian passes through that point. If the orders of the surfaces are all equal the point of common intersection is double on the Jacobian. If the orders of three of the surfaces are equal, the fourth touches the Jacobian. If the orders of two surfaces are equal, the line of intersection of the third and fourth touches the Jacobian.

(*c*) At a point common to the intersection of four surfaces of the same order m,

$$q\,.\,\mathrm{d}^2J=-m(m-1)\Sigma\pm[p_2p_3p_4]\mathrm{S}\mathrm{d}qf_1\mathrm{d}q,\ \text{where } \mathrm{d}p_n=(m-1)f_n\mathrm{d}q\,;$$

and hence the equation of the tangent cone at the double point is

$$\Sigma\pm(ap_2p_3p_4)\mathrm{S}rf_1r=0,$$

where a is an arbitrary constant quaternion.

(*d*) If four surfaces have a common multiple-point of order k, we find that

$$\mathrm{d}^{4k-3}\,.\,qJ=\Sigma\pm[\mathrm{d}^{k-1}p_2,\ \mathrm{d}^{k-1}p_3,\ \mathrm{d}^{k-1}p_4]\,.\,\mathrm{d}^kQ_1+\Sigma_0,$$
$$\mathrm{d}^{4k-4}\,.\,J=(\mathrm{d}^{k-1}p_1,\ \mathrm{d}^{k-1}p_2,\ \mathrm{d}^{k-1}p_3,\ \mathrm{d}^{k-1}p_4)+\Sigma_0',$$

where Σ_0 and Σ_0' denote sums of terms which vanish when q coincides with the multiple point, and we also have

$$\mathrm{d}^kQ_1=m_1\mathrm{S}\mathrm{d}q\mathrm{d}^{k-1}p_1+\text{vanishing terms.}$$

(*e*) At the multiple point $\mathrm{d}^{4k-5}J$ and $\mathrm{d}^{4k-4}\,.\,qJ$ vanish, and therefore $\mathrm{d}^{4k-4}J$ vanishes (as in (*b*)), and the Jacobian has a multiple point of order $4k-3$; and because we may write (as in (*a*))

$$\mathrm{d}^{4k-3}\,.\,qJ=m\mathrm{d}q\,.\,\mathrm{d}^{4k-4}J+\Sigma\pm(m_1-m)[\mathrm{d}^{k-1}p_2,\ \mathrm{d}^{k-1}p_3,\ \mathrm{d}^{k-1}p_4]\mathrm{S}\mathrm{d}q\mathrm{d}^{k-1}p_1+\Sigma_0'',$$

it follows when the surfaces are all of the same order that the Jacobian has a multiple point of order $4k-2$.

Ex. 3. Determine the equation of a surface which meets a given surface at the points of contact of lines which meet it in four consecutive points.

[This investigation, though rather long (compare *Three Dimensions*, pp. 559-567) affords some useful exercise in the manipulation of our formulae. If q is the point of contact and qr the tangent touching at four consecutive points, we have

$$Q=0,\ m\mathrm{S}rp=\mathrm{S}r\mathrm{D}\,.\,Q=0,\ m(m-1)\mathrm{S}rfr=\mathrm{S}r\mathrm{D}^2\,.\,Q=0,\ \mathrm{S}r\mathrm{D}^3\,.\,Q=0.$$

We may suppose the point r to lie in an arbitrary plane $\mathrm{S}rl=0$, and we have to obtain the resultant of the four equations in r and finally to free it from the arbitrary l. Let $\mathrm{S}ra=0$ and $\mathrm{S}rb=0$ be the equations of planes through the generators of the quadric (r variable) $\mathrm{S}rfr=0$ which lie in the tangent plane $\mathrm{S}rp=0$. Thus we have $r=[apl]$ and $r'=[bpl]$ for the points in which three generators meet the arbitrary plane. One or other of these points must lie on the cubic in r. Hence

$$\mathrm{S}r\mathrm{D}^3\,.\,Q\,.\,\mathrm{S}r'\mathrm{D}'^3\,.\,Q'=0,\ \text{or}\ \mathrm{S}r\mathrm{D}'^3\,.\,Q'\,.\,\mathrm{S}r'\mathrm{D}^3\,.\,Q=0,$$

or

$$(\mathrm{S}r\mathrm{D}^3\,.\,\mathrm{S}r'\mathrm{D}'^3+\mathrm{S}r'\mathrm{D}^3\,.\,\mathrm{S}r\mathrm{D}'^3)QQ'=0,$$

where the accents applied to D and Q are temporary marks connecting operator and operand. Now this may be written in the form

$$(4B^3-3ABC)QQ'=0,$$

where $\quad A=\mathrm{S}r\mathrm{DS}r'\mathrm{D},\ 2B=\mathrm{S}r\mathrm{D}'\mathrm{S}r'\mathrm{D}+\mathrm{S}r'\mathrm{DS}r\mathrm{D}',\ C=\mathrm{S}r\mathrm{D}'\mathrm{S}r'\mathrm{D}',$

and it is easy to express the operators A, B and C in terms of the function f.

In virtue of the definition of the planes $\mathrm{S}ra=0$, $\mathrm{S}rb=0$, we have *identically*

$$Srfr=\mathrm{S}ra\mathrm{S}rb+\mathrm{S}rp\mathrm{S}rc,$$

where $\mathrm{S}rc=0$ is some plane. Hence we find on replacing r and r' in A, B and C by $[apl]$ and $[bpl]$ that

$$A=\mathrm{S}[pl\mathrm{D}]f[pl\mathrm{D}],\quad B=\mathrm{S}[pl\mathrm{D}]f[pl\mathrm{D}'],\quad C=\mathrm{S}[pl\mathrm{D}']f[pl\mathrm{D}'].$$

Remembering that $p=fq$ and that $\mathrm{S}pq=0$, we have by Art. 146, Ex. 5,

$$B=-n\begin{vmatrix} 0 & \mathrm{S}ql & \mathrm{S}q\mathrm{D}' \\ \mathrm{S}ql & \mathrm{S}lf^{-1}l & \mathrm{S}lf^{-1}\mathrm{D}' \\ \mathrm{S}q\mathrm{D} & \mathrm{S}lf^{-1}\mathrm{D} & \mathrm{SD}f^{-1}\mathrm{D}' \end{vmatrix}=\mathrm{S}(\mathrm{DS}ql-l\mathrm{S}q\mathrm{D})F(\mathrm{D}'\mathrm{S}ql-l\mathrm{S}q\mathrm{D}'),$$

with similar expressions for A and C, where $F=nf^{-1}$ is Hamilton's auxiliary function. Writing for the moment $e=\mathrm{DS}ql-l\mathrm{S}q\mathrm{D}$ and remembering that D and D$'$ operate on Q and Q' solely and not on q as involved in the structure of the operators, we proceed to expand and to operate on Q'. We have

$$\begin{aligned}B^3Q'&=(\mathrm{S}eF\mathrm{D}'\,.\,\mathrm{S}ql-\mathrm{S}eFl\mathrm{S}q\mathrm{D}')^3\,.\,Q'\\&=\mathrm{S}eF\mathrm{D}'^3\,.\,Q'\,.\,\mathrm{S}ql^3-3m(m-1)(m-2)\,.\,n(\mathrm{S}eF e\mathrm{S}eFl\mathrm{S}ql^2-\mathrm{S}eFl^2\mathrm{S}eq\mathrm{S}ql),\end{aligned}$$

because by the identities at the beginning of this example

$$\mathrm{S}q\mathrm{D}'\,.\,\mathrm{S}eF\mathrm{D}'^2\,.\,Q'=m(m-1)(m-2)\mathrm{S}FefFe=m(m-1)(m-2)\,.\,n\,.\,\mathrm{S}eFe,$$

$$\mathrm{S}q\mathrm{D}'^2\,.\,\mathrm{S}eF\mathrm{D}'\,.\,Q'=m(m-1)(m-2)\mathrm{S}Fep=m(m-1)(m-2)\,.\,n\,.\,\mathrm{S}eq=0,$$

since $\qquad \mathrm{S}eq=0$ and $\mathrm{S}q\mathrm{D}'^3\,.\,Q'=m(m-1)(m-2)\,.\,Q=0.$

We retain for a purpose the term in $\mathrm{S}eq$.

In like manner

$$\begin{aligned}BC\,.\,Q'=\mathrm{SD}'F\mathrm{D}'\,.\,\mathrm{S}eF\mathrm{D}'\,.\,Q'\,.\,\mathrm{S}ql^3-\mathrm{S}q\mathrm{D}'(\mathrm{SD}'F\mathrm{D}'\,.\,Q'\mathrm{S}eFl\\+2\mathrm{S}lF\mathrm{D}'\mathrm{S}eF\mathrm{D}'\,.\,Q')\mathrm{S}ql^2+\mathrm{S}q\mathrm{D}'^2(\mathrm{S}eF\mathrm{D}'\,.\,Q'\mathrm{S}lFl+2\mathrm{S}lF\mathrm{D}'\,.\,Q'\,.\,\mathrm{S}eFl)\,.\,\mathrm{S}ql.\end{aligned}$$

The term $\mathrm{S}q\mathrm{D}'\,.\,\mathrm{SD}'F\mathrm{D}'\,.\,Q'$ may be reduced by writing for the moment $\mathrm{D}'=\Sigma a'\mathrm{S}a\mathrm{D}'$, where as is easily seen $\Sigma\mathrm{S}aa'=4$. This term becomes $m(m-1)(m-2)\Sigma\mathrm{S}afFa'=4m(m-1)(m-2)\,.\,n$, and hence we find

$$\begin{aligned}ABC\,.\,Q'&=\mathrm{S}eFe\,.\,\mathrm{S}eF\mathrm{D}'\,.\,\mathrm{SD}'F\mathrm{D}'\,.\,Q'\mathrm{S}ql^3\\&\quad-m(m-1)(m-2)n(4\mathrm{S}eFe\mathrm{S}eFl\mathrm{S}ql^2-\mathrm{S}eFe\mathrm{S}lFl\mathrm{S}eq\mathrm{S}ql).\end{aligned}$$

From these two relations we get, if $e'=\mathrm{D}'\mathrm{S}ql-l\mathrm{S}q\mathrm{D}'$,

$$\begin{aligned}(4B^3-3ABC)Q'&=(4\mathrm{S}eFe'^3-3\mathrm{S}eFe\mathrm{S}eFe'\mathrm{S}e'Fe')\,.\,Q'\\&=(4\mathrm{S}eF\mathrm{D}'^3-3\mathrm{S}eFe\mathrm{S}eF\mathrm{D}'\mathrm{SD}'F\mathrm{D}')\,.\,Q\,.\,\mathrm{S}ql^3\\&\qquad-3m(m-1)(m-2)\,.\,n\,.\,(\mathrm{S}eFe\mathrm{S}lFl-4\mathrm{S}eFl^2)\mathrm{S}eq\mathrm{S}ql,\end{aligned}$$

and the last term vanishes because $\mathrm{S}eq=0$. Now it will be observed that the operator in the first term is precisely the same as the original operator with D$'$ substituted for $\mathrm{D}'\mathrm{S}ql-l\mathrm{S}q\mathrm{D}'$. This remark allows us to write down the result of operating on QQ' in the form

$$\begin{aligned}(4B^3-3ABC)QQ'&=(4\mathrm{SD}F\mathrm{D}'^3-3\mathrm{SD}F\mathrm{DSD}F\mathrm{D}'\mathrm{SD}'F\mathrm{D}')\,.\,QQ'\,.\,\mathrm{S}ql^6\\&\quad-3m(m-1)(m-2)\,.\,n\,.\,(\mathrm{SD}'F\mathrm{D}'\mathrm{S}lFl-4\mathrm{SD}'Fl^2)\mathrm{S}q\mathrm{D}'\,.\,Q'\,.\,\mathrm{S}ql^4,\end{aligned}$$

the object of the retention of the term in Sqe being now apparent. But the term we have retained vanishes by the reduction we have already made use of. Thus Sql^6 comes off as a factor, and the equation of the surface is

$$(4SDFD'^3 - 3SDFDSDFD'SD'FD').QQ'=0.]$$

EXAMPLES TO CHAPTER XVII.

Ex. 1. A right line meets three fixed lines aa', bb' and cc'. The locus of the harmonic conjugate of the point of intersection on the third line with respect to the points on the other two is the intersection of the planes

$$(bb'cq)(aa'cc')+(aa\,cq)(bb'cc')=0\ ;\ (bb'c'q)(aa'cc')+(aa'c'q)(bb'cc')=0.$$

Ex. 2. The general equation of a quadric through the conic

$$Sqfq=0,\ Slq=0 \text{ is } Sqfq - SlqSl'q=0.$$

Find the value of l' in order that the quadric may be a cone having its vertex at a and show that the equation of the cone may be written in the form

$$S\{qSla - aSlq\}f\{qSla - aSlq\}=0.$$

Ex. 3. A plane $aa'p$ is drawn through a fixed line aa', and the lines in which it meets the planes $Slq=0$ and $Sl'q=0$ are joined to the points b and b' respectively. The equations of the joining planes are

$$(qaa'p)Slb-(baa'p)Slq=0 \text{ and } (qaa'p)Sl'b'-(b'aa'p)Sl'q=0,$$

respectively, and when p varies the locus of their intersection is the quadric surface

$$(qSlb-bSlq,\ qSl'b'-b'Sl'q,\ a,\ a')=0.$$

Ex. 4. The four faces of a tetrahedron pass each through a fixed point, a, b, c and d respectively. The three edges in the face p which contains the point d lie in the planes, l, m and n respectively. The vertex q opposite the face p is the intersection of the planes

$$SqlSap-SqpSal=0,\ SqmSbp-SqpSbm=0,\ SqnScp-SqpScn=0,$$

and the vertex q describes the cubic surface

$$(aSql-qSal,\ bSqn-qSbm,\ cSqn-qScn,\ d)=0,$$

having the intersection of the fixed planes as a double point.

Ex. 5. Find the locus of the vertex of a tetrahedron, if the three edges which pass through that vertex pass each through a fixed point, if the opposite face also passes through a fixed point and the three remaining vertices move in fixed planes.

Ex. 6. A plane passes through a fixed point d, and the points in which it meets three fixed lines a_1a_2, b_1b_2 and c_1c_2 are joined by planes to three other fixed lines a_3a_4, b_3b_4, and c_3c_4. The locus of intersection of the planes is the surface

$$(a_1(a_2a_3a_4q)-a_2(a_1a_3a_4q),\ b_1(b_2b_3b_4q)-b_2(b_1b_3b_4q),\ c_1(c_2c_3c_4q)-c_2(c_1c_3c_4q),\ d)=0.$$

Ex. 7. The sides of a polygon pass through fixed points, a_1, a_2, ... a_n, and all the vertices but one move in fixed planes, l_1, l_2, ... l_{n-1}. If q is the free vertex, the next is $f_1q=qSl_1a_1-a_1Sl_1q$, and the locus of the free vertex is the twisted cubic

$$[f_{n-1}f_{n-2}\cdots f_2f_1q,\ q,\ a_n]=0.$$

Ex. 8. All the sides of a polygon but one pass through fixed points a_1, a_2, ... a_{n-1}, the extremities of the free side move on fixed lines bb' and cc', and all the other vertices on fixed planes l_1, l_2, ... l_{n-2}; find the surface generated by the free side.

Ex. 9. The points of contact q of tangent planes through the line ab to the quadric $Sqfq=0$ satisfy the relation *

$$fq=x[abq], \text{ where } n+x^2\{S(fafb)(ab)-S[fafb][ab]\}=0,$$

n being the fourth invariant of f, and if c is arbitrary

$$q=[fa,\ fb,\ fc+x[abc]].$$

Ex. 10. If the line ab is a generator of the quadric $Sqfq=0$,

$$\frac{(fafb)}{[ab]}=-\frac{[fafb]}{(ab)}=\text{a scalar.}$$

Ex. 11. The generators of the family of quadrics $Sq(xf_1+yf_2+zf_3)q=0$ compose the complex of lines of the third order represented by the determinant equation

$$|Sqf_nq,\ Spf_nq,\ Spf_np|=0 \quad (n=1, 2 \text{ or } 3).$$

(a) When p is an arbitrarily selected fixed point, this equation represents a cubic cone, and every edge of the cone determines a definite quadric of the family. The tangent planes at p to the quadrics pass through the edge of the cone which joins p to the point $[f_1p, f_2p, f_3p]$; and the tangent plane to the cone along this edge touches at the point p the quadric of which the edge is a generator.

(b) When p lies on the Jacobian curve

$$[f_1p,\ f_2p,\ f_3p]=0,$$

the cubic cone breaks up into a plane and a quadric cone. The cone is a member of the family of quadrics, and the plane touches at p all the quadrics of the family which pass through p.

(c) The locus of points of contact of a plane $Slq=0$ with quadrics of the family is the cubic curve in which the plane cuts the surface

$$(l,\ f_1q,\ f_2q,\ f_3q)=0;$$

and the locus of points of contact of pairs of the quadrics is

$$[f_1q,\ f_2q,\ f_3q]=0.$$

Ex. 12. The integral of the differential equation

$$(dq,\ fq)=0, \text{ or } dq=fq\,.\,dt,$$

where f is a linear function, may be written in the form

$$q=e^{tf}\,.\,a,$$

where a is a quaternion constant of integration.

(a) This integral represents a doubly infinite family of curves, and a determinate curve of the family passes through an arbitrary point provided it is not a united point of the function f.

(b) The equation $\qquad p=e^{-tf'}\,.\,b$

is the reciprocal of the tangent line developable of the curve determined by a if the conditions

$$Sba=0,\quad Sbfa=0,\quad Sbf^2a=0$$

are satisfied.

(c) An arbitrary plane which does not pass through a united point of f is osculated by a single and determinate curve of the family.

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* For another form see Art. 146, Ex. 5.

(d) An arbitrary tangent line to an arbitrary curve of the family is cut in a constant anharmonic ratio at the point of contact and at the points of intersection with three of the united planes of f.

(e) A right line which cuts the faces of the tetrahedron in points having a certain anharmonic ratio touches a definite curve of the family, and if p and q are two points on the line

$$(p, q, fp, fq)=0.$$

(f) Any linear transformation which leaves unchanged the united points of f, merely interchanges curves of the family.

(g) The locus of points of contact of tangent lines drawn from an arbitrary point c to curves of the family is the twisted cubic

$$q=(f+u)^{-1}c\,;$$

the locus of points of contact of tangent lines drawn through an arbitrary line cd is the quadric

$$(cdqfq)=0\,;$$

and the locus of points of osculation of planes through c is the cubic surface

$$(c,\ q,\ fq,\ f^2q)=0.$$

Ex. 13. The equation of the complex of lines cutting a tetrahedron in points having a given anharmonic ratio may be written in the form

$$(p,\ q,\ fp,\ fq)=0 \quad \text{where} \quad \frac{t_1-t_2}{t_2-t_3}\cdot\frac{t_3-t_4}{t_4-t_1}=A$$

is the given anharmonic ratio, t_1, t_2, t_3 and t_4 being the latent roots of f and the tetrahedron being determined by the united points of the function.

(a) The differential equation of curves whose tangents cut the tetrahedron in points having the given anharmonic ratio is

$$(\mathrm{d}q,\ q,\ f\mathrm{d}q,\ fq)=0\,;$$

and a solution of this equation is

$$q=e^{\int\frac{f+u}{f+v}\,.\,\mathrm{d}t}\,.\,a,$$

where a is an arbitrary quaternion and where u and v are functions of t.

(b) This equation includes the family of curves (compare Ex. 10, p. 286)

$$q=(f+t)^m\,.\,a.$$

(c) In general the reciprocal of the tangent line developable of the curve (a) is

$$p=e^{-\int\frac{f'+u}{f'+v}\,.\,\mathrm{d}t}\,.\,(f'+v)^2\,.\,b,$$

where
$$\mathrm{S}ba=\mathrm{S}bfa=\mathrm{S}bf^2a=0.$$

(d) The anharmonic ratio of the point of contact and of the points in which a tangent line to the curve (a) cuts the faces of the tetrahedron corresponding to the roots t_1, t_2 and t_3 is

$$\frac{t_1+u}{t_3+u}\cdot\frac{t_3-t_2}{t_1-t_2}.$$

CHAPTER XVIII.

HYPERSPACE.

ART. **158.** Many of the methods of quaternions are applicable with but slight change to the general case of a "flat" space of n dimensions.

Commencing with the multiplication of two vectors or directed lines in space of n dimensions, we may suppose the two vectors to be transferred to one common plane or even to be made coinitial, and we may define the product $\alpha\beta$ very nearly in the same manner as in quaternions. In the formulae of definition

$$\alpha\beta = \mathrm{V}_2\alpha\beta + \mathrm{V}_0\alpha\beta, \quad \beta\alpha = -\mathrm{V}_2\alpha\beta + \mathrm{V}_0\alpha\beta, \qquad \text{(I.)}$$

$\mathrm{V}_0\alpha\beta$ or $\mathrm{S}\alpha\beta$ is minus the projection of one vector on the other multiplied by the length of the latter, and $\mathrm{V}_2\alpha\beta$ is the directed area of the parallelogram determined by α and β, rotation in the plane from α to β being positive. We can no longer identify $\mathrm{V}_2\alpha\beta$ with a vector perpendicular to the plane because in space of many dimensions there is an infinite number of directions perpendicular to a plane.

In particular if $i_1, i_2, \dots i_n$ are n mutually rectangular unit-vectors in the space of n dimensions, we have by (I.)

$$i_s^2 = -1, \quad i_t^2 = -1, \quad i_s i_t + i_t i_s = 0, \qquad \text{(II.)}$$

where s and t are any two numbers from 1 to n.

The functions $\mathrm{V}_2\alpha\beta$ and $\mathrm{V}_0\alpha\beta$ are doubly distributive, and hence the binary product $\alpha\beta$ is doubly distributive. We define for products of higher order that multiplication is thoroughly associative and distributive, and these principles in conjunction with (I.) form an adequate symbolical basis for the whole calculus.

If i_1 and i_2 are any two mutually rectangular unit vectors in the plane of α and β, and if rotation from i_1 to i_2 is in the same sense as that from α to β, we may write

$$\mathrm{V}_2\alpha\beta = i_1 i_2 \mathrm{T}\mathrm{V}_2\alpha\beta, \qquad \text{(III.)}$$

where $TV_2\alpha\beta$ is the number of units in the vector area $V_2\alpha\beta$. The symbol i_1i_2 represents a unit vector-area in the plane of $\alpha\beta$ or in any parallel plane. This symbol i_1i_2 is of a distinct kind from the symbols $i_1, i_2, \dots i_n$, and it cannot be expressed as a linear function of the latter.

In virtue of the laws of multiplication

$$i_1i_2 \,.\, i_1 = -i_2i_1 \,.\, i_1 = i_2 \quad\text{and}\quad i_1i_2 \,.\, i_2 = -i_1;$$

and hence by (III.) the effect of multiplying a vector area into a vector in its plane is to turn that vector through a right angle in the plane and to multiply its length by the number of units in the area.

For three vectors, which may be transferred to a common space of three dimensions or even rendered coinitial, the laws of the calculus allow us to write

$$\alpha\beta\gamma = V_3\alpha\beta\gamma + V_1\alpha\beta\gamma, \quad\dots\dots\dots\dots\dots\dots\text{(IV.)}$$

where $V_3\alpha\beta\gamma$ denotes the part of the product depending on sets of three distinct units combining in the irreducible products $i_1i_2i_3$, etc., and where $V_1\alpha\beta\gamma$ arises from reducible products such as $i_1{}^3 = -i_1$, $i_1{}^2i_2 = -i_2$, $i_1i_2i_1 = i_2$. In full if $\alpha = \Sigma x_1i_1$, $\beta = \Sigma y_1i_1$, $\gamma = \Sigma z_1i_1$, where x, y and z are scalar coefficients, we find

$$\left.\begin{aligned} V_3\alpha\beta\gamma &= \Sigma\,|x_1y_2z_3|\,i_1i_2i_3, \\ V_1\alpha\beta\gamma &= -\Sigma y_1z_1\Sigma x_1i_1 + \Sigma x_1z_1\Sigma y_1i_1 - \Sigma x_1y_1\Sigma z_1i_1, \end{aligned}\right\} \dots\text{(V.)}$$

where $|x_1y_2z_3|$ denotes a determinant.

The first part $V_3\alpha\beta\gamma$ of the product of three vectors represents the *directed volume* of the parallelepiped determined by the vectors, it being now necessary to distinguish between volumes in different spaces of three dimensions. In particular $i_1i_2i_3$ represents unit volume in the space of i_1, i_2 and i_3. The function $V_3\alpha\beta\gamma$ is evidently combinatorial with respect to the three vectors. It is unchanged when α is replaced by $\alpha + v\beta + w\gamma$, etc., and it changes sign when any two of the vectors are transposed.

We have given the expansion for $V_1\alpha\beta\gamma$ in terms of the unit vectors and of the scalars x, y, z; but there is another method of wide application which we may employ. It is apparent that we must have

$$V_1\alpha\beta\gamma = u\alpha V_0\beta\gamma + v\beta V_0\alpha\gamma + w\gamma V_0\alpha\beta,$$

where u, v and w are numbers. Interchanging β and γ we have

$$V_1\alpha\gamma\beta = u\alpha V_0\gamma\beta + v\gamma V_0\alpha\beta + w\beta V_0\alpha\gamma;$$

adding and attending to (I.) we find

$$V_1\alpha(\beta\gamma + \gamma\beta) = 2\alpha V_0\beta\gamma = 2u\alpha V_0\beta\gamma + (v+w)(\beta V_0\alpha\gamma + \gamma V_0\alpha\beta),$$

and thus $u=1$, $v+w=0$. Similarly interchanging α and β we find that $w=1$ and $u+v=0$, and thus

$$V_1\alpha\beta\gamma=\alpha V_0\beta\gamma-\beta V_0\alpha\gamma+\gamma V_0\alpha\beta. \qquad \text{(VI.)}$$

By the same process we arrive at the result

$$\alpha\beta\gamma\delta=V_4\alpha\beta\gamma\delta+V_2\alpha\beta\gamma\delta+V_0\alpha\beta\gamma\delta \qquad \text{(VII.)}$$

for the product of any four vectors, where

$$\left.\begin{aligned} V_2\alpha\beta\gamma\delta&=V_2\alpha\beta V_0\gamma\delta-V_2\alpha\gamma V_0\beta\delta+V_2\alpha\delta V_0\beta\gamma+V_2\beta\gamma V_0\alpha\delta\\ &\qquad -V_2\beta\delta V_0\alpha\gamma+V_2\gamma\delta V_0\alpha\beta;\\ V_0\alpha\beta\gamma\delta&=V_0\alpha\beta V_0\gamma\delta-V_0\alpha\gamma V_0\beta\delta+V_0\alpha\delta V_0\beta\gamma; \end{aligned}\right\} \text{(VIII.)}$$

and it will be noticed that in these relations the *determinant rule of signs* is in every case obeyed, namely starting with the term $\alpha V_0\beta\gamma$, the next term, in which β comes first, has a minus sign and so on. In like manner for five vectors

$$\left.\begin{aligned} &\alpha\beta\gamma\delta\epsilon=V_5\alpha\beta\gamma\delta\epsilon+V_3\alpha\beta\gamma\delta\epsilon+V_1\alpha\beta\gamma\delta\epsilon;\\ &V_3\alpha\beta\gamma\delta\epsilon=\Sigma\pm V_3\alpha\beta\gamma V_0\delta\epsilon;\ V_1\alpha\beta\gamma\delta\epsilon=\Sigma\pm\alpha V_0\beta\gamma\delta\epsilon; \end{aligned}\right\} \qquad \text{(IX.)}$$

the first terms in the sums being affected with the positive sign and the determinant law of signs being obeyed. (Compare Art. 147, p. 270.)

Considering more particularly the function of m vectors $V_m\alpha_1\alpha_2\ldots\alpha_m$, it is apparent from various points of view, that it is combinatorial with respect to the m vectors. We may prove that it changes sign whenever any two vectors are transposed, and hence we may deduce the combinatorial property. Adding the products

$$\left.\begin{aligned} \alpha_1\alpha_2\alpha_3\ldots\alpha_m&=(V_m+V_{m-2}+V_{m-4}+\text{etc.}\ldots)\alpha_1\alpha_2\alpha_3\ldots\alpha_m,\\ \alpha_2\alpha_1\alpha_3\ldots\alpha_m&=(V_m+V_{m-2}+V_{m-4}+\text{etc.}\ldots)\alpha_2\alpha_1\alpha_3\ldots\alpha_m, \end{aligned}\right\} \quad \text{(X.)}$$

in the second of which α_1 and α_2 are transposed, the sum is $2\alpha_3\alpha_4\ldots\alpha_m V_0\alpha_1\alpha_2$. In this sum the highest terms in the units are of the order $m-2$, and consequently interchange of contiguous vectors changes the sign of $V_m\alpha_1\alpha_2\ldots\alpha_p\ldots\alpha_m$. Hence transposition of any two vectors changes the sign; for example $p-1$ changes of sign accompany the transference of α_p to the first place in the function, and $p-2$ changes arise when α_1 is transposed with α_2, with α_3 and so on till it reaches the place originally occupied by α_p. The function consequently vanishes if any two vectors are identical, and when the vectors α are replaced by vectors β which are given as linear functions of the α, the function is simply multiplied by the modulus of the transformation.

Generally any function such as

$$\Sigma \pm V_p a_1 a_2 a_3 \ldots a_p V_{m-p} a_{p+1} a_{p+2} \ldots a_m$$

is combinatorial with respect to the vectors, and when we express the m vectors a in terms of m mutually perpendicular vector units in their m-space, we find that

$$\frac{\lfloor m}{\lfloor p \lfloor m-p} V_m a_1 a_2 \ldots a_m = \Sigma \pm V_p a_1 a_2 \ldots a_p V_{m-p} a_{p+1} a_{p+2} \ldots a_m. \quad \text{(XI.)}$$

This includes a number of relations such as

$$3V_3 a\beta\gamma = aV_2\beta\gamma - \beta V_2 a\gamma + \gamma V_2 a\beta;$$

$$aV_3\beta\gamma\delta - \beta V_3 a\gamma\delta + \gamma V_3 a\beta\delta - \delta V_3 a\beta\gamma = 0 \text{ if } V_4 a\beta\gamma\delta = 0.$$

Again when the m vectors lie in a space of $m-1$ dimensions so that they are linearly connected, we have relations of the form

$$a_m = \Sigma \frac{V_{m-2} a_2 a_3 \ldots a_{m-1} V_0 a_1 a_m}{V_{m-1} a_1 a_2 a_3 \ldots a_{m-1}}; \quad \text{(XII.)}$$

which may be verified by operating with $V_0 a_1$, etc. In particular for two and three dimensions, we recover the formulae, Art. 27 (III.) with $S\rho a\beta = 0$ and Art. 26 (II.).

The theory explained in this article may be compared with Grassmann's *Ausdehnungslehre*.* Grassmann's *inner product* of two quantities is the function $-V_0 a\beta$, and his *outer product* of $a_1, a_2, \ldots a_m$ is $V_m a_1 a_2 \ldots a_m$. These so-called products are thus only parts of a complete associative product.

ART. **159**. There is a remarkable difference between this general theory and the theory of quaternions which may be illustrated by a special example. *The sum of a number of vector areas is not an area vector*, or the homogeneous quadratic function of the units

$$A = V_2 aa' + V_2 \beta\beta' + V_2 \gamma\gamma' + \text{etc.} \quad \text{(I.)}$$

cannot generally be expressed in the form $V\rho\rho'$. The geometrical reason for this is that two planes, for example $\rho = x_1 i_1 + x_2 i_2$ and $\rho = x_3 i_3 + x_4 i_4$, have not necessarily a common line although they may have a common point—the origin of the vectors ρ in the example.

To discover a canonical system of vector units in terms of which a homogeneous function (q) of order m may be expressed, observe that $q\rho = V_{m+1} q\rho + V_{m-1} q\rho$, and that the line vector $V_1 q V_{m-1} q\rho$ is not generally parallel to ρ but that it is a linear and distributive function of ρ. We are thus led to consider the linear and distributive function

$$\phi\rho = V_1 q V_{m-1} q\rho, \quad \text{(II.)}$$

* See *Proc. R.I.A.*, 3rd Series, vol. vi., pp. 13–18 (1900).

and because

$$V_0\sigma\phi\rho = V_0 \,.\, \sigma q V_{m-1} q\rho = V_0 \,.\, V_{m-1}\sigma q V_{m-1} q\rho$$
$$= V_0 \,.\, V_{m-1} q\sigma V_{m-1}\rho q = V_0 \,.\, V_{m-1}\rho q V_{m-1} q\sigma = V_0\rho\phi\sigma,$$

the function ϕ is self-conjugate, and just as in quaternions its axes are all real and mutually rectangular.

In particular for the quadratic A, let i_1 be an axis of $\phi\rho = V_1 A V_1 A\rho$ so that $\phi i_1 = a_{12} i_1$, where a_{12} is a scalar.

Then $\phi V_1 A i_1 = V_1 A V_1 A V_1 A i = V_1 A \phi i_1 = a_{12} V_1 A i_1$, and $V_1 A i_1$ is also an axis and it is perpendicular to i_1 and parallel to i_2 suppose. This shows that in terms of the canonical units

$$A = a_{12} i_1 i_2 + a_{34} i_3 i_4 + \ldots + a_{2m-1,2m} i_{2m-1} i_{2m}, \quad \text{........(III.)}$$

so that a quadratic in $2m+1$ or in $2m$ units may be reduced to m terms involving pairs of units, or to the sum of m area vectors. There is obviously indeterminateness in the units to the extent that i_1 may be any unit in a definite plane—that of i_1 and i_2, and i_3 may be any unit in another definite plane, and so on.

An expression such as A corresponds to an *angular velocity* in the space of three dimensions. Consider the transformation which converts line vectors (ρ) into line vectors $(\sigma = \phi\rho)$ and which preserves unchanged lengths and mutual inclinations, so that

$$V_0\sigma\sigma' = V_0\phi\rho\phi\rho' = V_0\rho\rho'.$$

If a is an axis of this function and t the corresponding root, we have

$$t^2 a^2 = V_0\phi a^2 = V_0 a^2 = a^2,$$

and therefore $t = 1$ or else $a^2 = 0$. The former alternative corresponds to non-rotated directions. The latter requires a to be of the form $i_1 + \sqrt{-1} \,.\, i_2$—a vector perpendicular to itself directed to one of the circular points at infinity in the plane of i_1 and i_2 (Ex. 1, p. 96). Corresponding to this there is a conjugate axis, $a' = i_1 - \sqrt{-1}\, i_2$. Again if β is any other axis corresponding to the root s,

$$ts V_0 a\beta = V_0 \phi a \phi\beta = V_0 a\beta,$$

so that axes, corresponding to roots which are not reciprocals one of the other, must be perpendicular. From this it appears that the transformation is specially related to a set of hyperperpendicular planes, $i_1 i_2$, $i_3 i_4$, etc., and that it consists of ordinary rotations in each of these planes, so that we may write

$$\left.\begin{array}{l} \sigma = q\rho q^{-1}, \\ \text{where } q = q_{12} q_{34} q_{56} \cdots q_{2m-1,2m}, \quad q_{12} = \cos\tfrac{1}{2}a_{12} + i_1 i_2 \sin\tfrac{1}{2}a_{12}, \end{array}\right\} \ldots\text{(IV.)}$$

and where the factors q_{12}, q_{34}, etc., are commutative because we have

$$i_1 i_2 i_3 i_4 = i_3 i_1 i_2 i_4 = i_3 i_4 i_1 i_2.$$

Also we have $$i_1 i_2 \,.\, i_1 i_2 = i_2{}^2 = -1.$$

It may at once be verified that the operator $q_{12}(\quad)q_{12}{}^{-1}$ has no effect except on vectors in the plane of $i_1 i_2$, and that it turns vectors in this plane through the angle a_{12}.

Now we may write (Art. 29 (v.), p. 28)

$$q_{12} = (i_1 i_2)^{\frac{a_{12}}{\pi}} = e^{\frac{1}{2} i_1 i_2 a_{12}},$$

and because the factors are all commutative we may also write

$$q = e^{\frac{1}{2}(a_{12} i_1 i_2 + a_{34} i_3 i_4 + ..)} = e^{\frac{1}{2}A} \quad \text{.....................(v.)}$$

(compare (III.) and Art. 29 (x.)), and the rotation is effected by the operator $e^{\frac{1}{2}A}(\quad)e^{-\frac{1}{2}A}$.

For a small rotation, if $\mathrm{d}t$ is a small scalar whose square is to be neglected,

$$\sigma = e^{\frac{1}{2}A\mathrm{d}t} \rho e^{-\frac{1}{2}A\mathrm{d}t} = (1 + \tfrac{1}{2}A\,\mathrm{d}t)\rho(1 - \tfrac{1}{2}A\,\mathrm{d}t) = \rho + \mathrm{d}t\mathrm{V}_1 A\rho, \quad \text{...(VI.)}$$

and thus A plays the part of an angular velocity.*

ART. **160**. For projective geometry in n space we may use the method explained in the last chapter, and the symbol for a point is the sum of a scalar and a vector, so that

$$q = \mathrm{V}_0 q + \mathrm{V}_1 q = \left(1 + \frac{\mathrm{V}_1 q}{\mathrm{V}_0 q}\right)\mathrm{V}_0 q = (1 + \mathrm{OQ})\mathrm{V}_0 q \quad \text{.........(I.)}$$

represents a point of weight $\mathrm{V}_0 q$ at the extremity of the vector $\mathrm{V}_1 q : \mathrm{V}_0 q$.

The equation $$q = t_1 a_1 + t_2 a_2 + \text{etc.} \ldots + t_m a_m \quad \text{..................(II.)}$$

represents the $(m-1)$-flat which contains the points $a_1, a_2, \ldots a_m$.

In accordance with Hamilton's notation, we shall write

$$[a_1 a_2 \ldots a_m] = \mathrm{V}_m \,.\, \mathrm{V}_1 a_1 \mathrm{V}_1 a_2 \ldots \mathrm{V}_1 a_m$$
$$-\Sigma \pm \mathrm{V}_{m-1} \,.\, \mathrm{V}_1 a_2 \mathrm{V}_1 a_3 \ldots \mathrm{V}_1 a_m \mathrm{V}_0 a_1;$$

or briefly, $$[a]_m = \mathrm{V}_m [a]_m + \mathrm{V}_{m-1} [a]_m, \quad \text{.................(III.)}$$

as the symbol of the $(m-1)$-flat containing the m points a. To show that this symbol really determines the flat, observe that we have

$$[a]_m = \{\mathrm{V}_m a_1 a_2 \ldots a_m - \mathrm{V}_{m-1}(a_2 - a_1)(a_3 - a_1)\ldots(a_m - a_1)\}\Pi\mathrm{V}_0 a_1, \text{(IV.)}$$

where $a_1 \mathrm{V}_0 a_1 = \mathrm{V}_1 a_1$ and where $\Pi\mathrm{V}_0 a_1$ is the product of the weights of the points (Art. 144 (IV.)). Now $\mathrm{V}_m a_1 a_2 \ldots a_m$ or $\mathrm{V}_m a_1(a_2 - a_1) \ldots (a_m - a_1)$ is the directed region determined by the origin and the m points, and $\mathrm{V}_{m-1}(a_2 - a_1)(a_3 - a_1) \ldots (a_m - a_1)$ is

* See *Proc. R.I.A.*, Series III., vol. v., pp. 73–123.

that determined by the m points. Denoting the latter by $i_1 i_2 \dots i_{m-1} R$, where R is the magnitude of the region determined by the m points and where $i_1 i_2 \dots i_{m-1}$ are mutually perpendicular unit vectors in the flat, the symbol becomes

$$[a]_m = (\varpi - 1) i_1 i_2 \dots i_{m-1} \,.\, R \Pi V_0 a_1, \text{ ....(v.)}$$

where ϖ is the component of the vector a_1 which is perpendicular to every line in the flat, or in other words, where ϖ is the vector perpendicular from the origin to the flat. But when we know ϖ and the product of the vectors i we know the flat,* and we have

$$\varpi = -\frac{V_m[a]_m}{V_{m-1}[a]_m} \quad \text{and} \quad i_1 i_2 \dots i_{m-1} = -UV_{m-1}[a]_m, \text{ ....(vi.)}$$

where U has its quaternion signification. We notice also that the point

$$P_m = 1 + \frac{V_{m-1}[a]_m}{V_m[a]_m} = 1 - \varpi^{-1} \text{ ....(vii.)}$$

is the reciprocal with respect to the auxiliary quadric $V_0 q^2 = 0$ of every point in the flat—in other words, this point is the point in the m-flat of the origin and of the m points a which is reciprocal to the $(m-1)$-flat of the points a.

In point symbols the equation of the flat is

$$[q a_1 a_2 \dots a_m] = 0, \text{ ....(viii.)}$$

the vanishing of this equation being equivalent to (ii.).

Other general expressions admit easily of interpretation on the principles laid down in this article.

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* The vector equation of the flat is $\rho = \varpi + \Sigma x_1 i_1$.

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* See Linear vector function, the use of an ellipsoid being to a great extent superseded.

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